

# Hardy spaces, Regularized BMO spaces and the boundedness of Calderón-Zygmund operators on non-homogeneous spaces

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## Abstract

One defines a non-homogeneous space  $(X, \mu)$  as a metric space equipped with a non-doubling measure  $\mu$  so that the volume of the ball with center  $x$ , radius  $r$  has an upper bound of the form  $r^n$  for some  $n > 0$ . The aim of this paper is to study the boundedness of Calderón-Zygmund singular integral operators  $T$  on various function spaces on  $(X, \mu)$  such as the Hardy spaces, the  $L^p$  spaces and the regularized BMO spaces. This article thus extends the work of X. Tolsa [T1] on the non-homogeneous space  $(\mathbb{R}^n, \mu)$  to the setting of a general non-homogeneous space  $(X, \mu)$ . Our framework of the non-homogeneous space  $(X, \mu)$  is similar to that of [Hy] and we are able to obtain quite a few properties similar to those of Calderón-Zygmund operators on doubling spaces such as the weak type  $(1, 1)$  estimate, boundedness from Hardy space into  $L^1$ , boundedness from  $L^\infty$  into the regularized BMO and an interpolation theorem. Furthermore, we prove that the dual space of the Hardy space is the regularized BMO space, obtain a Calderón-Zygmund decomposition on the non-homogeneous space  $(X, \mu)$  and use this decomposition to show the boundedness of the maximal operators in the form of Cotlar inequality as well as the boundedness of commutators of Calderón-Zygmund operators and BMO functions.

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# 1 Introduction

In the last few decades, Calderón-Zygmund theory of singular integrals has played a central part of modern harmonic analysis with lots of extensive applications to other fields of mathematics. This theory has established criteria for singular integral operators to be bounded on various function spaces including  $L^p$  spaces,  $1 < p < \infty$ , Hardy spaces, BMO spaces and Besov spaces.

One of the main features of the standard Calderón-Zygmund singular integral theory is the requirement that the underlying spaces or domains to possess the doubling (volume) property. Recall that a space  $X$  equipped with a distance  $d$  and a measure  $\mu$  is said to have the doubling property if there exists a constant  $C$  such that for all  $x \in X$  and all  $r > 0$ ,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r))$$

where  $B(x, r)$  denotes the ball with center  $x$  and radius  $r > 0$ .

In the last ten years or so, there has been substantial progress in obtaining boundedness of singular integrals acting on spaces without the doubling property. Many features of the standard Calderón-Zygmund singular integral theory was extended to spaces with a mild volume growth condition in place of doubling property through the works of Nazarov, Treil, Volberg, Tolsa and others. See, for example [NTV1], [NTV2], [NTV3], [T1] and [T2]. These breakthroughs disproved the long held belief of the decades of 70's and 80's that the doubling property is indispensable in the theory of Calderón-Zygmund singular

integrals and lead to more powerful techniques and estimates in harmonic analysis.

Let  $X$  be a metric space equipped with a measure  $\mu$ , possibly non-doubling, satisfying

$$\mu(B(x, r)) \leq Cr^n$$

for some positive constants  $C$ ,  $n$  and all  $r > 0$ . We will call such a space  $(X, \mu)$  a non-homogeneous space. Let  $T$  be a Calderón-Zygmund operator acting on a non-homogeneous space  $X$ , i.e. the associated kernel of  $T$  satisfies appropriate bounds and has Hölder continuity (for the precise definition, see Section 2.1). Assume that  $T$  is bounded on  $L^2(X)$ , then it is shown in [NTV2] that the Calderón-Zygmund operator  $T$  is of weak type  $(1, 1)$ , hence by interpolation, is bounded on  $L^p(X)$ ,  $1 < p < \infty$ . See also [T2].

Hardy spaces and BMO spaces on a non-homogeneous space  $X$  were studied by a number of authors, for example [T1], [MMNO], [Hy]. In [MMNO], the authors studied the spaces  $\text{BMO}(\mu)$  and  $H_{at}^1(\mu)$  on  $\mathbb{R}^n$  (with  $\text{BMO}(\mu)$  space being defined via the standard bounded oscillations and the Hardy space  $H_{at}^1(\mu)$  being defined by an atomic decomposition) for a non doubling measure  $\mu$  and showed some standard properties of these spaces such as the John-Nirenberg inequality, an interpolation theorem between  $\text{BMO}(\mu)$  and  $H_{at}^1(\mu)$ , and  $\text{BMO}(\mu)$  being the dual space of  $H_{at}^1(\mu)$ . However, it is shown by Verdera [V] that an  $L^2$  bounded Calderón-Zygmund operator may be unbounded from  $L^\infty(\mu)$  into  $\text{BMO}(\mu)$  as well as from  $H_{at}^1(\mu)$  into  $L^1(\mu)$ . This shows the need to introduce variants of the BMO spaces characterized by bounded oscillation estimates so that the Calderón-Zygmund operators are bounded from  $L^\infty(\mu)$  into these variants of BMO spaces.

In [T1], the author introduced the RBMO space, a variant of the space BMO, on the non-homogeneous space  $(\mathbb{R}^n, \mu)$  which retains some of the properties of the standard BMO such as the John-Nirenberg inequality. See Section 3 for the definition of RBMO spaces. While Calderón-Zygmund operators might not be bounded from  $L^\infty(\mathbb{R}^n, \mu)$  into  $\text{BMO}(\mathbb{R}^n, \mu)$ , they are bounded from  $L^\infty(\mathbb{R}^n, \mu)$  into  $\text{RBMO}(\mathbb{R}^n, \mu)$ , [T1].

Recently, Hytönen studied the RBMO spaces on non-homogeneous spaces  $(X, \mu)$  (instead of  $(\mathbb{R}^n, \mu)$ ) [Hy]. He proved that the space  $\text{RBMO}(\mu)$  on  $X$  still satisfies John-Nirenberg inequality. However, the boundedness of Calderón-Zygmund operators from  $L^\infty(\mu)$  into  $\text{RBMO}(\mu)$  and a number of other properties are still open questions for the setting of general non-homogeneous spaces  $(X, \mu)$ .

In this article, our aim is to conduct an extensive study on the RBMO spaces on general non-homogeneous spaces. More specifically, for a non-homogeneous space  $(X, \mu)$  equipped with a measure  $\mu$  which is dominated by some doubling measure (the same setting as in [Hy]), we are able to prove the following new results:

- (i) An  $L^2$  bounded Calderón-Zygmund operator is bounded from  $L^\infty(\mu)$  into the RBMO space, see Theorem 7.1.

- (ii) The dual space of the atomic Hardy spaces is shown to be the RBMO space. We also show that an  $L^2$  bounded Calderón-Zygmund operator is bounded from the atomic Hardy space  $H_{at}^1(\mu)$  into  $L^1(\mu)$ , see Theorems 5.6 and 7.3.
- (iii) An interpolation theorem between the RBMO space and the Hardy space  $H_{at}^1(\mu)$ : if an operator is bounded from  $H_{at}^1(\mu)$  into  $L^1(\mu)$  and from  $L^\infty(\mu)$  into the RBMO space, then the operator is bounded on  $L^p(\mu)$  for all  $1 < p < \infty$ , see Theorem 6.4.
- (iv) A Cotlar type inequality for Calderón-Zygmund operators which gives the boundedness of several maximal operators associated with  $T$ , see Theorem 6.6.
- (v) The boundedness of commutators of Calderón-Zygmund operators and RBMO functions on  $L^p$  spaces, see Theorem 7.6.
- (vi) A Calderón-Zygmund decomposition using a variant of Vitali covering lemma, see Theorem 6.3, and the weak type  $(1, 1)$  of an  $L^2$  bounded Calderón-Zygmund operator, see Theorem 6.5.

We remark that, while this manuscript was in finishing touch, we learned that similar results concerning the Hardy spaces as in (ii) have been obtained independently in [HyYY].

We now give a brief comment about some techniques used in this paper. In addition to using some ideas and techniques in [T1], we obtain certain key estimates through careful investigation of the family of doubling balls in a non-homogeneous space  $(X, \mu)$ . Let us recall that the main techniques used in [T1] rely on the Besicovitch covering lemma and the construction of the  $(\alpha, \beta)$ -doubling balls in  $\mathbb{R}^n$ . However, the Besicovitch covering lemma is only applicable to  $\mathbb{R}^n$  and it is not applicable in the setting of general non-homogeneous spaces. In the general setting, one can construction the  $(\alpha, \beta)$ -doubling balls by using a covering lemma in [He] in place of the Besicovitch covering lemma. In [Hy], the author used this substitution to obtain the John-Nirenberg inequality for the RBMO spaces. However, it seems that to obtain further results similar to the standard theory as in the case of doubling spaces, more refined techniques are needed.

An important technical detail in this paper is our construction of the three consecutive  $(\alpha, \beta)$ -doubling balls (see, Proposition 2.4) which we employ successfully to obtain the important characterizations (9) and (10), similar to those in [T1, Lemma 2.10]. By using these three consecutive  $(\alpha, \beta)$ -doubling balls, we show the boundedness of Calderón-Zygmund operators from  $L^\infty(X, \mu)$  into the space RBMO (see Theorem 7.1) as well as an interpolation theorem of RBMO spaces (see Theorem 4.3).

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## 2 Non-homogeneous spaces, families of doubling balls and singular integrals

### 2.1 Non-homogeneous spaces and families of doubling balls

In this paper, for the sake of simplicity we always assume that  $(X, d)$  is a metric space. With minor modifications, similar results hold when  $X$  is a quasi-metric space.

**Geometrically doubling regular metric spaces.** We adopt the definition that the space  $(X, d)$  is geometrically doubling if there exists a number  $N \in \mathbb{N}$  such that every open ball  $B(x, r) = \{y \in X : d(y, x) < r\}$  can be covered by at most  $N$  balls of radius  $r/2$ . Our using of this somewhat non-standard name is to differentiate this property from other types of doubling properties. If there is no specification, the ball  $B$  means the ball center  $x_B$  with radius  $r_B$ . Also, we set  $n = \log_2 N$ , which can be viewed as (an upper bound for) a geometric dimension of the space. Let us recall the following well-known lemma. See, for example [Hy].

**Lemma 2.1** *In a geometrically doubling regular metric space, a ball  $B(x, r)$  can contain the centers  $x_i$  of at most  $N\alpha^{-n}$  disjoint balls  $B(x_i, \alpha r)$  for any  $\alpha \in (0, 1]$ .*

**Upper doubling measures.** A measure  $\mu$  in the metric space  $(X, \mu)$  is said to be an upper doubling measure if there exists a dominating function  $\lambda$  with the following properties:

- (i)  $\lambda : X \times (0, \infty) \mapsto (0, \infty)$ ;
- (ii) for any fixed  $x \in X$ ,  $r \mapsto \lambda(x, r)$  is increasing;
- (iii) there exists a constant  $C_\lambda > 0$  such that  $\lambda(x, 2r) \leq C_\lambda \lambda(x, r)$  for all  $x \in X$ ,  $r > 0$ ;
- (iv) the inequality  $\mu(x, r) := \mu(B(x, r)) \leq \lambda(x, r)$  holds for all  $x \in X$ ,  $r > 0$ ;
- (v) and  $\lambda(x, r) \approx \lambda(y, r)$  for all  $r > 0$ ,  $x, y \in X$  and  $d(x, y) \leq r$ .

We note that in [Hy], the condition (v) is not assumed.

**Lemma 2.2** *Every family of balls  $\{B_i\}_{i \in F}$  of uniformly bounded diameter in a metric space  $X$  contains a disjoint sub-family  $\{B_i\}_{i \in E}$  with  $E \subset F$  such that*

$$\cup_{i \in F} B_i \subset \cup_{i \in E} 5B_i.$$

For a proof of Lemma 2.2, see [He].

**Assumptions:** Throughout the paper, we always assume that  $(X, \mu)$  is a geometrically doubling regular metric space and the measure  $\mu$  is an upper doubling measure.

We adopt the following definition as in [T1]. For  $\alpha, \beta > 1$ , a ball  $B \subset X$  is called  $(\alpha, \beta)$ -doubling if  $\mu(\alpha B) \leq \beta \mu(B)$ . The following result states the existence of plenty of doubling balls with small radii and with large radii.

**Lemma 2.3** ([Hy]) *The following statements hold:*

- (i) *If  $\beta > C_\lambda^{\log_2 \alpha}$ , then for any ball  $B \subset X$  there exists  $j \in \mathbb{N}$  such that  $\alpha^j B$  is  $(\alpha, \beta)$ -doubling.*
- (ii) *If  $\beta > \alpha^n$  where  $n$  is the doubling order of  $\lambda$ , then for any ball  $B \subset X$  there exists  $j \in \mathbb{N}$  such that  $\alpha^{-j} B$  is  $(\alpha, \beta)$ -doubling.*

Our following result which shows the existence of three consecutive  $(\alpha, \beta)$  doubling balls will play an important role in this paper.

**Proposition 2.4** *If  $B$  is a  $(\alpha^3, \beta)$  doubling ball ( $\alpha > 1$ ), then  $B, \alpha B$  and  $\alpha^2 B$  are three consecutive  $(\alpha, \beta)$  doubling balls.*

*Proof:* The proof of Proposition 2.4 is simple, hence we omit the details here.

For any two balls  $B \subset Q$ , we defined

$$K_{B,Q} = 1 + \int_{r_B \leq d(x, x_B) \leq r_Q} \frac{1}{\lambda(x_B, d(x, x_B))} d\mu(x). \quad (1)$$

This definition is a variant of the definition in [T1, pp.94-95]. Similarly to the results [T1, Lemma 2.1] we have the following properties:

**Lemma 2.5** (i) *If  $Q \subset R \subset S$  are balls in  $X$ , then*

$$\max\{K_{Q,R}, K_{R,S}\} \leq K_{Q,S} \leq C(K_{Q,R} + K_{R,S}).$$

(ii) *If  $Q \subset R$  are compatible size, then  $K_{Q,R} \leq C$ .*

(iii) *If  $\alpha Q, \dots, \alpha^{N-1} Q$  are non  $(\alpha, \beta)$ -doubling balls ( $\beta > C_\lambda^{\log_2 \alpha}$ ) then  $K_{Q, \alpha^N Q} \leq C$ .*

The proof of Lemma 2.5 is not difficult, hence we omit the details here.

As in [T1], for two balls  $B \subset Q$  we can define the coefficient  $K'_{B,Q}$  as follows: let  $N_{B,Q}$  be the smallest integer satisfying  $6^{N_{B,Q}} r_B \geq r_Q$ , then we set

$$K'_{B,Q} := 1 + \sum_{k=1}^{N_{B,Q}} \frac{\mu(6^k B)}{\lambda(x_B, 6^k r_B)}.$$

In the case that  $\lambda(x, ar) = a^m \lambda(x, r)$  for all  $x \in X$  and  $a, r > 0$ , it is not difficult to show that  $K_{B,Q} \approx K'_{B,Q}$ . However, in general, we only have  $K_{B,Q} \leq C K'_{B,Q}$ .

## 2.2 Calderón-Zygmund operators

A kernel  $K(\cdot, \cdot) \in L^1_{\text{loc}}(X \times X \setminus \{(x, y) : x = y\})$  is called a Calderón-Zygmund kernel if

(i)

$$|K(x, y)| \leq C \min \left\{ \frac{1}{\lambda(x, d(x, y))}, \frac{1}{\lambda(y, d(x, y))} \right\}. \quad (2)$$

(ii) There exists  $0 < \delta \leq 1$  such that

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{d(x, x')^\delta}{d(x, y)^\delta \lambda(x, d(x, y))} \quad (3)$$

if  $d(x, x') \leq Cd(x, y)$ .

A linear operator  $T$  is called a Calderón-Zygmund operator with kernel  $K(\cdot, \cdot)$  satisfying (2) and (3) if for all  $f \in L^\infty(\mu)$  with bounded support and  $x \notin \text{supp} f$ ,

$$Tf(x) = \int_X K(x, y)f(y)d\mu(y).$$

The maximal operator  $T_*$  associated with the Calderón-Zygmund operator  $T$  is defined by

$$T_*f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)|,$$

where  $T_\epsilon f(x) = \int_{d(x, y) \geq \epsilon} K(x, y)f(y)d\mu(y)$ .

We would like to give an example for the operator whose the associated kernel satisfies the conditions (2) and (3). As in [Hy], we consider Bergman-type operators which are studied by Volberg and Wick. In [VW], the authors obtained a characterization of measures  $\mu$  in the unit ball  $\mathbb{B}_{2n}$  of  $\mathbb{C}^n$  for which the analytic Besov-Sobolev space  $B_2^\sigma(\mathbb{B}^{2n})$  embeds continuously into  $L^2(\mu)$ . Their proof goes through a new  $T1$  theorem for what they call Bergman-type operators. Let us describe the situation of this application. The measures  $\mu$  in [VW] satisfy the upper power bound  $\mu(B(x, r)) \leq r^m$ , except possibly when  $B(x, r) \subset H$ , where  $H$  is a fixed open set. However, in the exceptional case there holds  $r \leq \delta(x) := d(x, H^c)$ , and hence

$$\mu(B(x, r)) \leq \lim_{\epsilon \rightarrow 0} \mu(B(x, \delta(x) + \epsilon)) \leq \lim_{\epsilon \rightarrow 0} (\delta(x) + \epsilon)^m = \delta^m.$$

Thus we find that their measures are actually upper doubling with

$$\mu(B(x, r)) \leq \max\{\delta(x)^m, r^m\} =: \lambda(x, r).$$

It is not difficult to show that  $\lambda(\cdot, \cdot)$  satisfies the conditions (i)-(v) in definition of upper doubling measures.

In [VW], as a main application concerning the Besov-Sobolev spaces, the authors introduced the operator associated to the kernel

$$K(x, y) = (1 - \bar{x} \cdot y)^{-m}, \quad (4)$$

for  $x, y \in \overline{\mathbb{B}}_{2n} \subset \mathbb{C}^n$ . Here  $\overline{x}$  stands for the componentwise complex conjugation, and  $\overline{x} \cdot y$  designates the usual dot product of  $n$ -vectors  $\overline{x}$  and  $y$ . Moreover, one equips  $\overline{\mathbb{B}}_{2n}$  with the regular quasi-distance, see [Tch, Lemma 2.6],

$$d(x, y) := \left| |x| - |y| \right| + \left| 1 - \frac{\overline{x} \cdot y}{|x||y|} \right|.$$

Finally, the set  $H$  related to the exceptional balls is now the open unit ball  $\overline{\mathbb{B}}_{2n}$ . It was proved in [HyM] that the kernel  $K(x, y)$  defined by (4) satisfies (2) and (3).

### 3 The RBMO spaces

#### 3.1 Definition of $\text{RBMO}(\mu)$

The RBMO (Regularized BMO) space was introduced by Tolsa for  $(\mathbb{R}^n, \mu)$  in [T1] and it was adopted by T. Hytönen for general non-homogeneous space  $(X, \mu)$  in [Hy].

**Definition 3.1** Fix a parameter  $\rho > 1$ . A function  $f \in L^1_{\text{loc}}(\mu)$  is said to be in the space  $\text{RBMO}(\mu)$  if there exists a number  $C$ , and for every ball  $B$ , a number  $f_B$  such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| d\mu(x) \leq A \quad (5)$$

and, for any two balls  $B$  and  $B_1$  such that  $B \subset B_1$ ,

$$|f_B - f_{B_1}| \leq CK_{B, B_1}. \quad (6)$$

The infimum of the values  $C$  in (6) is taken to be the RBMO norm of  $f$  and denoted by  $\|f\|_{\text{RBMO}(\mu)}$ .

The RBMO norm  $\|\cdot\|_{\text{RBMO}(\mu)}$  is independent of  $\rho > 1$ . Moreover the John-Nirenberg inequality holds for  $\text{RBMO}(X)$ . More precisely, we have the following result (see Corollary 6.3 in [Hy]).

**Proposition 3.2** For any  $\rho > 1$  and  $p \in [1, \infty)$ , there exists a constant  $C$  so that for every  $f \in \text{RBMO}(\mu)$  and every ball  $B_0$ ,

$$\left( \frac{1}{\mu(\rho B_0)} \int_{B_0} |f(x) - f_{B_0}|^p d\mu(x) \right)^{1/p} \leq C \|f\|_{\text{RBMO}(\mu)}.$$

#### 3.2 Some characterizations of $\text{RBMO}(\mu)$

In the rest of paper, unless  $\alpha$  and  $\beta$  are specified otherwise, by an  $(\alpha, \beta)$  doubling ball we mean a  $(6, \beta_0)$ -doubling with a fixed number  $\beta_0 > \max\{C_\lambda^{3 \log_2 6}, 6^{3n}\}$ .

Given a ball  $B \subset X$ , let  $N$  be the smallest non-negative integer such that  $\tilde{B} = 6^N B$  is doubling. Such a ball  $\tilde{B}$  exists due to Lemma 2.5.



Let  $\rho > 1$  be some fixed constant. We say that  $f \in L^1_{\text{loc}}(\mu)$  is in  $\text{RBMO}(\mu)$  if there exists some constant  $C > 0$  such that for any ball  $Q$

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - m_{\tilde{B}} f| d\mu(x) \leq C \quad (7)$$

and

$$|m_Q f - m_R f| \leq C K_{Q,R}, \quad \text{for any two doubling balls } Q \subset R, \quad (8)$$

here  $m_B f$  is the mean value of  $f$  over the ball  $B$ . Then we take

$$\|f\|_* := \inf\{C : (7) \text{ and } (8) \text{ hold}\}.$$

By the same proof as in Lemma 2.8 of [T1], we have the following result.

**Proposition 3.3** *For a fixed  $\rho > 1$ , the norms  $\|\cdot\|_*$  and  $\|\cdot\|_{\text{RBMO}(\mu)}$  are equivalent.*

We now extend certain characterizations of  $\text{RBMO}(\mu)$  in [T1] in the case of  $(\mathbb{R}^n, \mu)$  to the case of non-homogeneous spaces  $(X, \mu)$ . In the case of  $\mathbb{R}^n$ , Besicovitch covering lemma was used but this lemma is not applicable in our setting. We overcome this problem by using the three consecutive doubling balls in Proposition 2.4.

**Proposition 3.4** *For  $f \in L^1_{\text{loc}}(\mu)$ , the following are equivalent:*

(a)  $f \in \text{RBMO}(\mu)$ .

(b) *There exists some constant  $C_b$  such that for any ball  $B$*

$$\frac{1}{\mu(6B)} \int_B |f(x) - m_B f| d\mu(x) \leq C_b \quad (9)$$

and

$$|m_Q f - m_R f| \leq C_b K_{Q,R} \left( \frac{\mu(6Q)}{\mu(Q)} + \frac{\mu(6R)}{\mu(R)} \right), \quad \text{for any two balls } Q \subset R. \quad (10)$$

(c) *There exists some constant  $C_c$  such that for any doubling ball  $B$*

$$\frac{1}{\mu(B)} \int_B |f(x) - m_B f| d\mu(x) \leq C_c \quad (11)$$

and

$$|m_Q f - m_R f| \leq C_c K_{Q,R}, \quad \text{for any two doubling balls } Q \subset R. \quad (12)$$

Moreover, the best constants  $C_b$  and  $C_c$  are comparable to the  $\text{RBMO}(\mu)$  norm of  $f$ .

*Proof:* (a)  $\rightarrow$  (b) : If  $f \in \text{RBMO}(\mu)$ , then (9) and (10) hold for  $C_b = C\|f\|_*$  for some constant  $C$ . Indeed, for any ball  $B$  we have

$$|m_B f - m_{\tilde{B}} f| \leq m_Q(|f - m_{\tilde{B}} f|) \leq \|f\|_* \frac{\mu(6Q)}{Q}.$$

Therefore,

$$\frac{1}{\mu(6B)} \int_B |f(x) - m_B f| d\mu(x) \leq \frac{1}{\mu(6B)} \int_B (|f - m_{\tilde{B}} f| + |m_B f - m_{\tilde{B}} f|) \leq 2\|f\|_*. \quad (13)$$

On the other hand, for any two balls  $Q \subset R$ , one has

$$|m_Q f - m_R f| \leq |m_Q f - m_{\tilde{Q}} f| + |m_{\tilde{Q}} f - m_{\tilde{R}} f| + |m_R f - m_{\tilde{R}} f|.$$

Applying (13) for the first and the third terms, we have

$$|m_Q f - m_{\tilde{Q}} f| + |m_R f - m_{\tilde{R}} f| \leq \|f\|_* \left( \frac{\mu(6Q)}{Q} + \frac{\mu(6R)}{R} \right).$$

We can follow the argument in [T1] to obtain the estimate for the second term. Let us remark that for any two balls  $Q \subset R$  such that  $\tilde{Q} \subset \tilde{R}$ , it follows from (8) that

$$|m_{\tilde{Q}} f - m_{\tilde{R}} f| \leq \|f\|_* K_{\tilde{Q}, \tilde{R}}.$$

By Lemma 2.5, we have

$$K_{\tilde{Q}, \tilde{R}} \leq C(K_{Q, \tilde{Q}} + K_{Q, R} + K_{R, \tilde{R}}) \leq C(C_1 + K_{Q, R} + C_2) \leq CK_{Q, R}.$$

In general,  $Q \subset R$  does not imply  $\tilde{Q} \subset \tilde{R}$ . We consider two cases:

*Case 1:* If  $r_{\tilde{Q}} \geq r_{\tilde{R}}$ , then  $\tilde{Q} \subset 3\tilde{R}$ . Setting  $R_0 = 3\tilde{R}$ , then it follows from Lemma 2.5 and (8) that

$$\begin{aligned} |m_{\tilde{Q}} f - m_{\tilde{R}} f| &\leq |m_{\tilde{Q}} f - m_{R_0} f| + |m_{R_0} f - m_{\tilde{R}} f| \\ &\leq (K_{\tilde{Q}, R_0} + K_{\tilde{R}, R_0}) \|f\|_*. \end{aligned}$$

For the term  $K_{\tilde{Q}, R_0}$  we have

$$\begin{aligned} K_{\tilde{Q}, R_0} &\leq CK_{Q, R_0} \\ &\leq C(K_{Q, R} + K_{R, R_0}) \\ &\leq C(K_{Q, R} + K_{R, \tilde{R}} + K_{\tilde{R}, 3\tilde{R}} + K_{3\tilde{R}, R_0}) \\ &\leq CK_{Q, R}. \end{aligned}$$

The remaining term  $K_{\tilde{R}, R_0}$  is dominated by

$$C(K_{\tilde{R}, 3\tilde{R}} + K_{3\tilde{R}, R_0}) \leq CK_{Q, R}.$$

So in this case, we obtain  $|m_{\tilde{Q}} f - m_{\tilde{R}} f| \leq CK_{Q, R} \|f\|_*$ .

*Case 2:* If  $r_{\tilde{R}} < r_{\tilde{Q}}$ , then  $\tilde{R} \subset 6^2\tilde{Q}$ . Obviously, we can find some  $m \geq 1$  such that  $r_{\tilde{R}} \geq \frac{r_{5^m Q}}{25}$  and  $\tilde{R} \subset 6^m Q \subset 6^2\tilde{Q}$ . Therefore,  $\tilde{R}$  and  $5^m Q$  are comparable sizes. This implies  $K_{\tilde{R}, 5^m Q} \leq C$ . Setting  $Q_0 = \widetilde{6^2\tilde{Q}}$  we have

$$\begin{aligned} |m_{\tilde{Q}}f - m_{\tilde{R}}f| &\leq |m_{\tilde{Q}}f - m_{Q_0}f| + |m_{Q_0}f - m_{\tilde{R}}f| \\ &\leq (K_{\tilde{Q}, Q_0} + K_{\tilde{R}, Q_0})\|f\|_*. \end{aligned}$$

Let us estimate  $K_{\tilde{Q}, Q_0}$ . We have

$$K_{\tilde{Q}, Q_0} \leq C(K_{\tilde{Q}, 6^2\tilde{Q}} + K_{6^2\tilde{Q}, Q_0}) \leq CK_{Q, R}.$$

For the term  $K_{\tilde{R}, Q_0}$ , one has

$$\begin{aligned} K_{\tilde{R}, Q_0} &\leq C(K_{\tilde{R}, 5^m Q} + K_{5^m Q, 6^2\tilde{Q}} + K_{6^2\tilde{Q}, Q_0}) \\ &\leq C(K_{\tilde{R}, 5^m Q} + K_{Q, 6^2\tilde{Q}} + K_{6^2\tilde{Q}, Q_0}) \\ &\leq CK_{Q, R}. \end{aligned}$$

Therefore, in this case we also obtain  $|m_{\tilde{Q}}f - m_{\tilde{R}}f| \leq CK_{Q, R}\|f\|_*$ .

(b)  $\rightarrow$  (c): the proof of this implication is easy and hence we omit the detail here.

(c)  $\rightarrow$  (a): Let  $B$  be some ball. We need to show that (7) holds for  $\rho = 6$ . For any  $x \in B$ , there exists some  $(6^3, \beta_0)$ -doubling ball centered  $x$  with radius  $r_{6^{-2j}B}$  for some  $j \in \mathbb{N}$ . We denote by  $B_x$  the biggest ball satisfying these properties. Let us recall that by Proposition 2.4, the balls  $B_x, 6B_x$  and  $6^2B_x$  are three  $(6, \beta_0)$ -doubling balls. Moreover, by Lemma 2.5 we have

$$|m_{6B_x}f - m_{\tilde{B}}f| \leq |m_{6B_x}f - m_{B_x}f| + |m_{B_x}f - m_{\tilde{B}}f| \leq CC_c.$$

By Lemma 2.2, we can pick a disjoint subcollection  $B_{x_i}, i \in I$ , such that  $B \subset \cup_{i \in I} 5B_{x_i} \subset \cup_{i \in I} 6B_{x_i}$ . Thus, we have

$$\begin{aligned} \int_B |f - m_{\tilde{B}}f| d\mu &\leq \sum_{i \in I} \int_{B_{x_i}} |f - m_{\tilde{B}}f| d\mu \\ &\leq \sum_{i \in I} \int_{B_{x_i}} (|f - m_{6B_{x_i}}f| + |m_{6B_{x_i}}f - m_{\tilde{B}}f|) d\mu \\ &\leq \sum_{i \in I} CC_c \mu(6B_{x_i}) \\ &\leq \sum_{i \in I} C\beta_0 C_c \mu(B_{x_i}) \\ &\leq C\beta_0 C_c \mu(6B). \end{aligned}$$

This completes our proof.

## 4 Interpolation results

### 4.1 The sharp maximal operator

Adapting an idea in [T1], we define the sharp maximal operator as follows:

$$M^\sharp f(x) = \sup_{B \ni x} \frac{1}{\mu(6B)} \int_B |f - m_{\tilde{B}} f| d\mu + \sup_{(Q,R) \in \Delta_x} \frac{|m_Q f - m_R f|}{K_{Q,R}}, \quad (14)$$

here  $\Delta_x := \{(Q, R) : x \in Q \subset R \text{ and } Q, R : \text{doubling}\}$ .

Note that in our sharp maximal operator, the term  $\mu(6B)$  was chosen with the fixed constant 6 throughout the paper. It is clear that

$$f \in \text{RBMO}(\mu) \Leftrightarrow M^\sharp f \in L^\infty(\mu).$$

We define, for  $\rho \geq 1$ , the non-centered maximal operator  $M_{(\rho)}$  by setting

$$M_{(\rho)} f(x) = \sup_{x \in Q} \frac{1}{\mu(\rho Q)} \int_Q |f| d\mu.$$

It was proved that  $M_{(\rho)}$  is of weak type  $(1, 1)$  for  $\rho \geq 5$  and hence  $M_{(\rho)}$  is bounded on  $L^p(\mu)$  for all  $p \in (1, \infty]$ , see [Hy, Proposition 3.5]. When  $\rho = 1$ , we write  $Mf$  instead of  $M_{(1)}f$ . From the boundedness of  $M_{(\rho)}$  for  $\rho \geq 5$ , the non-centered doubling maximal operator is defined by

$$Nf(x) = \sup_{x \in Q: \text{doubling}} \frac{1}{\mu(Q)} \int_Q |f| d\mu$$

where the supremum is taken over all  $(6, \beta_0)$  doubling balls, is of weak type  $(1, 1)$  and hence bounded on  $L^p(\mu)$  for all  $p \in (1, \infty]$ .

Note that it is not difficult to show that

$$M^\sharp f(x) \leq M_{(6)} f(x) + 3Nf(x)$$

for all  $x \in X$ . Therefore the operator  $M^\sharp$  is of type weak  $(1, 1)$  and bounded on  $L^p(\mu)$  for all  $1 < p < \infty$ .

**Lemma 4.1** *For  $f \in L^1_{\text{loc}}(\mu)$ , we have*

$$M^\sharp |f|(x) \leq 5\beta_0 M^\sharp f(x).$$

The proof is similar to that of Remark 6.1 in [T1].

We now show that the non-centered doubling maximal operator is dominated by the sharp maximal operator in the following theorem. Although, some estimates are inspired from [T1, Theorem 6.2], there are some main differences in our proof. More specifically, the three consecutive doubling balls argument will be used to replace the Besicovitch covering lemma.

**Theorem 4.2** *Let  $f \in L^1_{\text{loc}}(\mu)$  with the extra condition  $\int f d\mu = 0$  if  $\|\mu\| := \mu(X) < \infty$ . Assume that for some  $p$ ,  $1 < p < \infty$ ,  $\inf\{1, Nf\} \in L^p(\mu)$ . Then we have*

$$\|Nf\|_{L^p(\mu)} \leq C\|M^\sharp f\|_{L^p(\mu)}.$$

*Proof:* We assume that  $\|\mu\| = \infty$ . The proof for  $\|\mu\| < \infty$  is similar. By standard argument, it suffices to prove the following  $\lambda$ -good inequality: for some fixed  $\nu < 1$  and all  $\epsilon > 0$  there exists some  $\delta > 0$  such that for any  $\lambda > 0$  we have

$$\mu\{x : Nf(x) > (1 + \epsilon)\lambda, M^\sharp f(x) \leq \delta\lambda\} \leq \nu\mu\{x : Nf(x) > \lambda\}. \quad (15)$$

Setting  $E_\lambda = \{x : Nf(x) > (1 + \epsilon)\lambda, M^\sharp f(x) \leq \delta\lambda\}$  and  $\Omega_\lambda = \{x : Nf(x) > \lambda\}$ , for  $f \in L^p(\mu)$ . For each  $x \in E_\lambda$ , we can choose the doubling ball  $Q_x$  containing  $x$  satisfying that  $m_{Q_x}|f| > (1 + \epsilon/2)\lambda$  and if  $Q$  is any doubling ball containing  $x$  with  $r_Q > 2r_{Q_x}$  then  $m_Q|f| \leq (1 + \epsilon/2)\lambda$ . Such a ball  $Q_x$  exists due to  $f \in L^p(\mu)$ .

Let  $R_x$  be the ball centered  $x$  with radius  $6r_{Q_x}$  and  $S_x$  be the smallest  $(6^3, \beta_0)$ -doubling ball in the form  $6^{3j}R_x$ . Then, by Proposition 2.4,  $S_x, 6S_x$  and  $6^2S_x$  are three  $(6, \beta_0)$ -doubling balls. Moreover, one has

$$K_{Q_x, 6S_x} \leq C(K_{Q_x, R_x} + K_{R_x, S_x} + K_{S_x, 6S_x}) \leq C.$$

Therefore, it follows from Lemma 4.1 that

$$|m_{Q_x}|f| - m_{6S_x}|f|| \leq K_{Q_x, 6S_x} M^\sharp |f|(x) \leq C\beta_0 M^\sharp f(x) \leq C\beta_0 \delta \lambda.$$

This implies that for sufficiently small  $\delta$  we have

$$m_{6S_x}|f| > \lambda$$

and hence  $6S_x \subset \Omega_\lambda$ .

Note that by Lemma 2.2, we can pick a disjoint collection  $\{S_{x_i}\}_{i \in I}$  with  $x_i \in E_\lambda$  and  $E_\lambda \subset \cup_{i \in I} 5S_{x_i} \subset \cup_{i \in I} 6S_{x_i}$ . Setting  $W_{x_i} = 6S_{x_i}$ , we will show that

$$\mu(6S_{x_i} \cap E_\lambda) \leq C \frac{\nu}{\beta_0} \mu(6S_{x_i}) \quad (16)$$

for all  $i \in I$ .

Once (16) is proved, (15) follows readily. Indeed, from (16) we have

$$\mu(E_\lambda) \leq \sum_{i \in I} \mu(6S_{x_i} \cap E_\lambda) \leq \sum_{i \in I} \frac{\nu}{\beta_0} \mu(6S_{x_i}) \leq C \sum_{i \in I} \nu \mu(S_{x_i}) \leq C\nu \mu(\Omega_\lambda).$$

Now we show the proof of (16). Let  $y \in W_{x_i} \cap E_\lambda$ . For any doubling ball  $Q \ni y$  satisfying  $m_Q|f| > (1 + \epsilon)\lambda$ , it follows that  $r_Q \leq r_{W_{x_i}}/8$ . Indeed, if  $r_Q > r_{W_{x_i}}/8$  then we have  $Q_{x_i} \subset W_{x_i} \subset \widetilde{16Q}$  and

$$|m_Q|f| - m_{\widetilde{16Q}}|f|| \leq K_{Q, \widetilde{16Q}} M^\sharp |f|(y) \leq C\delta\lambda \leq \frac{\epsilon}{2}$$

for sufficiently small  $\delta$ . This implies  $m_{16Q} \widetilde{|f|} > (1 + \epsilon/2)\lambda$  which is a contradiction to the choice of  $Q_{x_i}$ . So,  $r_Q \leq r_{W_{x_i}}/8$ . This, together with  $m_Q |f| > (1 + \epsilon)\lambda$ , imply

$$N(f\chi_{\frac{5}{4}W_{x_i}})(y) > (1 + \epsilon)\lambda$$

and

$$m_{\frac{5}{4}W_{x_i}} \widetilde{|f|} \leq (1 + \epsilon/2)\lambda \text{ (since } r_{\frac{5}{4}W_{x_i}} > 2r_{Q_{x_i}}).$$

This yields,

$$N(\chi_{\frac{5}{4}W_{x_i}} |f| - m_{\frac{5}{4}W_{x_i}} \widetilde{|f|})(y) > \frac{\epsilon}{2}\lambda.$$

Therefore, by using the weak  $(1, 1)$  boundedness of  $N$ , we have

$$\begin{aligned} \mu(W_{x_i} \cap E_\lambda) &\leq \mu\{y : N(\chi_{\frac{5}{4}W_{x_i}} |f| - m_{\frac{5}{4}W_{x_i}} \widetilde{|f|})(y) > \frac{\epsilon}{2}\lambda\} \\ &\leq \frac{C}{\epsilon\lambda} \int_{\frac{5}{4}W_{x_i}} (|f| - m_{\frac{5}{4}W_{x_i}} \widetilde{|f|}) d\mu \\ &\leq \frac{C}{\epsilon\lambda} \mu\left(\frac{15}{2}W_{x_i}\right) M^\sharp |f|(x_i) \\ &\leq \frac{C\delta}{\epsilon} \beta_0 \mu(6^3 S_{x_i}) \\ &\leq \frac{C\delta}{\epsilon} \beta_0 \mu(S_{x_i}). \end{aligned}$$

Thus, (16) holds provided  $\delta < \epsilon/C\nu\beta_0$ .

For the case  $f \notin L^p(\mu)$ , we define the sequence of functions  $\{f_k\}$ ,  $k = 1, 2, \dots$  by setting

$$f_k(x) = \begin{cases} f(x), & |f(x)| \leq k, \\ k \frac{f(x)}{|f(x)|}, & |f(x)| > k. \end{cases}$$

Then we have  $M^\sharp f_k(x) \leq CM^\sharp f(x)$ . On the other hand,  $|f_k(x)| \leq k \inf\{1, |f(x)|\} \leq k \inf(1, Nf)(x)$  and so  $f_k \in L^p(\mu)$ . Hence,

$$\|Nf_k\|_{L^p(\mu)} \leq C\|M^\sharp f_k\|_{L^p(\mu)} \leq C\|M^\sharp f\|_{L^p(\mu)}.$$

Taking the limit as  $k \rightarrow \infty$ , we obtain the required result and the proof is completed.

## 4.2 An Interpolation Theorem for linear operators

**Theorem 4.3** *Let  $1 < p < \infty$  and let  $T$  be a linear operator bounded on  $L^p(\mu)$  and from  $L^\infty(\mu)$  into  $\text{RBMO}(\mu)$ . Then  $T$  extends to a bounded operator on  $L^r(\mu)$  for  $p < r < \infty$ .*

*Proof:* We consider 2 cases:

**Case 1:**  $\|\mu\| = \infty$ : Since  $T$  is bounded on  $L^p(\mu)$ ,  $M^\sharp T$  is sublinear bounded on  $L^p(\mu)$  and on  $L^\infty(\mu)$ . Therefore, by interpolation,  $M^\sharp T$  is bounded on  $L^r(\mu)$  for  $p < r < \infty$ ,

$$\|M^\sharp T f\|_{L^r(\mu)} \leq C\|f\|_{L^r(\mu)}.$$

Assume that  $f \in L^r(\mu)$  is supported in compact set. Then  $f \in L^p(\mu)$  and so  $Tf \in L^p(\mu)$ . Hence  $Nf \in L^p(\mu)$  and  $\inf\{1, Nf\} \in L^r(\mu)$ . By invoking Theorem 4.2,

$$\|Tf\|_{L^r(\mu)} \leq \|M^\sharp Tf\|_{L^r(\mu)} \leq C\|f\|_{L^r(\mu)}.$$

**Case 2:** Assume that  $\|\mu\| < \infty$ . For  $f \in L^r(\mu)$ , set  $f = (f - \int f d\mu) + \int f d\mu = f_1 + f_2$ . Since  $\int f_1 d\mu = 0$ , we can apply the same argument as for  $\|\mu\| = \infty$ . It is not difficult to show that  $\|T1\|_{L^r(\mu)} \leq C\|1\|_{L^r(\mu)}$ . This completes the proof.

## 5 Atomic Hardy spaces and their dual spaces

### 5.1 The space $H_{at}^{1,\infty}(\mu)$

For a fixed  $\rho > 1$ , a function  $b \in L_{loc}^1(\mu)$  is called an atomic block if

- (i) there exists some ball  $B$  such that  $\text{supp } b \subset B$ ;
- (ii)  $\int b d\mu = 0$ ;
- (iii) there are functions  $a_j$  supported on cubes  $B_j \subset B$  and numbers  $\lambda_j \in \mathbb{R}$  such that

$$b = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{17}$$

where the sum converges in  $L^1(\mu)$ , and  $\|a_j\|_{L^\infty(\mu)} \leq (\mu(\rho B_j) K_{B_j, B})^{-1}$  and the constant  $K_{B_j, B}$  being given in the paragraph before Lemma 2.5.

We denote  $|b|_{H_{at}^{1,\infty}(\mu)} = \sum_{j=1}^{\infty} |\lambda_j|$ . We say that  $f \in H_{at}^{1,\infty}(\mu)$  if there are atomic blocks  $b_i$  such that

$$f = \sum_{i=1}^{\infty} b_i \tag{18}$$

with  $\sum_{i=1}^{\infty} |b_i|_{H_{at}^{1,\infty}(\mu)} < \infty$ . The  $H_{at}^{1,\infty}(\mu)$  norm of  $f$  is defined by

$$\|f\|_{H_{at}^{1,\infty}(\mu)} := \inf \sum_{i=1}^{\infty} |b_i|_{H_{at}^{1,\infty}(\mu)}$$

where the infimum is taken over all the possible decompositions of  $f$  in atomic blocks.

We have the following basic properties of  $H_{at}^{1,\infty}(\mu)$ .

**Proposition 5.1** (a)  $H_{at}^{1,\infty}(\mu)$  is a Banach space.

(b)  $H_{at}^{1,\infty}(\mu) \subset L^1(\mu)$  and  $\|f\|_{L^1(\mu)} \leq \|f\|_{H_{at}^{1,\infty}(\mu)}$ .

(c) The space  $H_{at}^{1,\infty}(\mu)$  is independent of the constant  $\rho$  when  $\rho > 1$ .

*Proof:* The proofs of (a) and (b) are standard and we omit the details here.

The proof of (c): Given  $\rho_1 > \rho_2 > 0$ , it is clear that  $H_{at,\rho_1}^{1,\infty} \subset H_{at,\rho_2}^{1,\infty}$  with  $\|f\|_{H_{at,\rho_2}^{1,\infty}} \leq \|f\|_{H_{at,\rho_1}^{1,\infty}}$ . Conversely, if  $b = \sum_{i=1}^{\infty} \lambda_i a_i$  is an atomic block with  $\text{supp } a_i \subset B_i \subset B$  in  $H_{at,\rho_1}^{1,\infty}$ , then by Lemma 2.1 we can cover each  $B_i$  by  $N \left[ \frac{\rho_1}{\rho_2} \right]^n$  balls, says  $\{B_{ik}\}$ , with the same radius  $\frac{\rho_2}{\rho_1} r_B$ . Therefore, we can decompose  $a_i := \sum_k a_{ik}$  where  $a_{ik} := a_i \frac{\chi_{B_{ik}}}{\sum_j \chi_{B_{ij}}}$ . It is not difficult to verify that  $b$  is also an atomic block in  $H_{at,\rho_2}^{1,\infty}$ . This completes our proof.

We now show that the space  $\text{RBMO}(\mu)$  is embedded in the dual space of  $H_{at}^{1,\infty}(\mu)$ .

**Lemma 5.2** *We have*

$$\text{RBMO}(\mu) \subset H_{at}^{1,\infty}(\mu)^*.$$

*That is, for  $g \in \text{RBMO}(\mu)$ , the linear functional*

$$L_g(f) = \int_X f g d\mu$$

*defines a continuous linear functional  $L_g$  over  $H_{at}^{1,\infty}(\mu)$  with  $\|L_g\|_{H_{at}^{1,\infty}(\mu)^*} \leq C \|g\|_{\text{RBMO}(\mu)}$ .*

*Proof:* Following standard argument, see for example [CW2, p.64], we only need to check that for an atomic block  $b$  and  $g \in \text{RBMO}(\mu)$ , we have

$$\left| \int b g d\mu \right| \leq C |b|_{H_{at}^{1,\infty}(\mu)} \|g\|_{\text{RBMO}(\mu)}.$$

Assume that  $\text{supp } b \subset B$  and  $b = \sum_j^{\infty} \lambda_j a_j$ , where  $a_j$ 's are functions satisfying (a) and (b) in the definition of atomic blocks. If  $g \in L^\infty$ , by using  $\int b d\mu = 0$ , we have

$$\left| \int b g d\mu \right| = \left| \int b (g - g_B) d\mu \right| \leq \sum_j^{\infty} |\lambda_j| \|a_j\|_{L^\infty(\mu)} \int_{B_i} |g - g_B| d\mu. \quad (19)$$

Since  $g \in L^\infty(\mu) \subset \text{RBMO}(\mu)$ , we have

$$\begin{aligned} \int_{B_i} |g - g_B| d\mu &\leq \int_{B_i} |g - g_{B_i}| d\mu + \int_{B_i} |g_B - g_{B_i}| d\mu \\ &\leq C K_{B_i, B} \|g\|_{\text{RBMO}(\mu)} \mu(\rho B_j). \end{aligned}$$

From (19), we obtain

$$\left| \int b g d\mu \right| \leq C |b|_{H_{at}^{1,\infty}(\mu)} \|g\|_{\text{RBMO}(\mu)}.$$

In general case, if  $g \in \text{RBMO}(\mu)$ , define

$$g_N(x) := \begin{cases} f(x), & |f(x)| < N, \\ N \frac{f(x)}{|f(x)|}, & |f(x)| \geq N. \end{cases}$$



It can be verified that  $\|g_N\|_{\text{RBMO}(\mu)} \leq C\|g\|_{\text{RBMO}(\mu)}$ . As above, since  $g_N \in L^\infty(\mu)$ , we have

$$\left| \int f g_N d\mu \right| \leq C\|f\|_{H_{at}^{1,\infty}(\mu)}\|g_N\|_{\text{RBMO}(\mu)} \leq C\|f\|_{H_{at}^{1,\infty}(\mu)}\|g\|_{\text{RBMO}(\mu)}.$$

Let us denote  $L_0^\infty := \{f : f \text{ in } L^\infty(\mu) \text{ with bounded support}\}$  and  $D = H_{at}^{1,\infty}(\mu) \cap L_0^\infty$ . So, the functional  $L_g : f \mapsto \int gf$  is well-defined on  $D$  whenever  $g \in \text{RBMO}(\mu)$  (since  $g \in L_{loc}^1(\mu)$ ). By the dominated convergence theorem

$$\lim_{N \rightarrow \infty} \int f g_N d\mu = \int f g d\mu$$

for all  $f \in D$ . We claim that  $D$  is dense in  $H_{at}^{1,\infty}(\mu)$ . To verify this claim, denote by  $H_{at,fin}^{1,\infty}(\mu)$  the set of all elements in  $H_{at}^{1,\infty}(\mu)$  where the sums (17) and (18) are taken over finite elements. Obviously,  $H_{at,fin}^{1,\infty}(\mu)$  is dense in  $H_{at}^{1,\infty}(\mu)$  and each functional  $f \in H_{at,fin}^{1,\infty}(\mu)$  is also in  $L_0^\infty$ . Therefore,  $L_b$  is a unique extension on  $H_{at}^{1,\infty}(\mu)$  and hence

$$\left| \int f g d\mu \right| \leq C\|f\|_{H_{at}^{1,\infty}(\mu)}\|g\|_{\text{RBMO}(\mu)}.$$

This completes our proof.

The following lemma can be obtained by the same argument as in [T1, Lemma 4.4].

**Lemma 5.3** *If  $g \in \text{RBMO}(\mu)$ , we have*

$$\|L_g\|_{H_{at}^{1,\infty}(\mu)} \approx \|g\|_{\text{RBMO}(\mu)}.$$

## 5.2 The space $H_{at}^{1,p}(\mu)$

For a fixed  $\rho > 1$ , a function  $b \in L_{loc}^1(\mu)$  is called a  $p$ -atomic block,  $1 < p < \infty$ , if

- (i) there exists some ball  $B$  such that  $\text{supp } b \subset B$ ;
- (ii)  $\int b d\mu = 0$ ;
- (iii) there are functions  $a_j$  supported on cubes  $B_j \subset B$  and numbers  $\lambda_j \in \mathbb{R}$  such that

$$b = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{20}$$

where the sum converges in  $L^1(\mu)$ , and

$$\|a_j\|_{L^p(\mu)} \leq (\mu(\rho B_j))^{1/p-1} K_{B_j, B}^{-1}.$$

We denote  $|b|_{H_{at}^{1,p}(\mu)} = \sum_{j=1}^{\infty} |\lambda_j|$ . We say that  $f \in H_{at}^{1,p}(\mu)$  if there are  $p$ -atomic blocks  $b_i$  such that

$$f = \sum_{i=1}^{\infty} b_i \tag{21}$$

with  $\sum_{i=1}^{\infty} |b_i|_{H_{at}^{1,p}(\mu)} < \infty$ . The  $H_{at}^{1,p}(\mu)$  norm of  $f$  is defined by

$$\|f\|_{H_{at}^{1,p}(\mu)} := \inf \sum_{i=1}^{\infty} |b_i|_{H_{at}^{1,p}(\mu)}$$

where the infimum is taken over all the possible decompositions of  $f$  in  $p$ -atomic blocks.

Similarly to  $H_{at}^{1,\infty}(\mu)$ , we have the following basic properties of  $H_{at}^{1,p}(\mu)$

**Proposition 5.4** (a)  $H_{at}^{1,p}(\mu)$  is a Banach space.

(b)  $H_{at}^{1,p}(\mu) \subset L^1(\mu)$  and  $\|f\|_{L^1(\mu)} \leq \|f\|_{H_{at}^{1,p}(\mu)}$ .

(c) The space  $H_{at}^{1,p}(\mu)$  is independent of the constant  $\rho$  when  $\rho > 1$ .

The proofs of this proposition is in line with Proposition 5.1, so we omit the details here.

**Lemma 5.5** We have

$$\text{RBMO}(\mu) \subset H_{at}^{1,p}(\mu)^*.$$

That is, for  $g \in \text{RBMO}(\mu)$ , the linear functional

$$L_g(f) = \int_X f g d\mu$$

defines a continuous linear functional  $L_g$  over  $H_{at}^{1,p}(\mu)$  with

$$\|L_g\|_{H_{at}^{1,p}(\mu)^*} \leq C \|g\|_{\text{RBMO}(\mu)}.$$

*Proof:* The proof of this lemma is analogous to that of Lemma 5.2 with minor modifications. We leave the details to the interested reader.

We remark that a main difference between the Hardy space in Tolsa's setting [T1] and our Hardy space in this article is the sense of convergence in the atomic decomposition. This leads to different approaches in proving the inclusions  $\text{RBMO}(\mu) \subset H_{at}^{1,\infty}(\mu)^*$  and  $\text{RBMO}(\mu) \subset H_{at}^{1,p}(\mu)^*$ . However, for the inverse inclusion  $H_{at}^{1,p}(\mu)^* \subset \text{RBMO}(\mu)$ , by a careful investigation, Tolsa [T1] showed that one only needs to consider the sums in (20) and (21) over finite  $p$ -atoms and  $p$ -atomic blocks, hence the sense of convergence in (20) and (21) does not matter in both settings. This is the reason why we can use the arguments in [T1] for our setting with minor modifications to obtain the duality result of  $H_{at}^{1,\infty}(\mu)$  and  $H_{at}^{1,p}(\mu)$  as in the next Theorem.

**Theorem 5.6** For  $1 < p < \infty$ ,  $H_{at}^{1,p}(\mu) = H_{at}^{1,\infty}(\mu)$ . Also  $H_{at}^{1,p}(\mu)^* = H_{at}^{1,\infty}(\mu)^* = \text{RBMO}(\mu)$ .

As explained above, we omit the details of the proof.

## 6 Calderón-Zygmund decomposition

### 6.1 Calderón-Zygmund decomposition

The following two technical lemmas will be useful for the construction of a Calderón-Zygmund decomposition on non-homogeneous spaces.

**Lemma 6.1** *Assume that  $Q, S$  are two concentric balls,  $Q \subset R$ , such that there are no  $(\alpha, \beta)$ -doubling balls with  $\beta > C_\lambda^{\log_2 \alpha}$  in the form  $\alpha^k Q, k \in \mathbb{N}$  such that  $Q \subset \alpha^k Q \subset R$ . Then we have*

$$\int_{R \setminus Q} \frac{1}{\lambda(x_Q, d(x_Q, x))} d\mu(x) \leq C.$$

*Proof:* Let  $N$  be the smallest integer such that  $R \subset \alpha^N Q$ . Then,  $\mu(\alpha^k Q) \geq \beta \mu(\alpha^{k-1} Q)$  for all  $k = 1, \dots, N$ . Therefore, we have,

$$\begin{aligned} & \int_{R \setminus Q} \frac{1}{\lambda(x_Q, d(x_Q, x))} d\mu(x) \\ & \leq \sum_{k=1}^N \int_{\alpha^{k-1} r_Q \leq d(x, y) \leq \alpha^k r_Q} \frac{1}{\lambda(x_Q, d(x_Q, x))} d\mu(x) \\ & \leq \sum_{k=1}^N \frac{\mu(\alpha^k Q)}{\lambda(x_Q, \alpha^{k-1} r_Q)} \\ & \leq \sum_{k=1}^N \frac{\beta^{N-k} \mu(\alpha^N Q)}{(C_\lambda)^{(N-k) \log_2 \alpha} \lambda(x_Q, \alpha^N r_Q)} \\ & \leq \sum_{k=1}^N \left[ \frac{\beta}{(C_\lambda)^{\log_2 \alpha}} \right]^{N-k} \\ & \leq \sum_{j=1}^{\infty} \left[ \frac{\beta}{(C_\lambda)^{\log_2 \alpha}} \right]^j \\ & \leq C \quad (\text{since } \beta > \log_2 \alpha). \end{aligned}$$

This completes the proof.

While the Covering Lemma 2.2 for  $(X, \mu)$  can be used to replace the Besicovich covering lemma for  $(\mathbb{R}^n, \mu)$  in certain estimates, the Calderón-Zygmund decomposition in  $(X, \mu)$  will need a covering lemma which gives the finite overlapping property at all points  $x \in X$ . This is given in the next lemma.

**Lemma 6.2** *Every family of balls  $\{B_i\}_{i \in F}$  of uniformly bounded diameter in a metric space  $X$  contains a disjoint sub-family  $\{B_i\}_{i \in E}$  with  $E \subset F$  such that*

$$(i) \quad \cup_{i \in F} B_i \subset \cup_{i \in E} 6B_i,$$

$$(ii) \quad \text{For each } x \in X, \sum_{i \in E} \chi_{6B_i} < \infty.$$

We remark that in (ii), the sum  $\sum_{i \in E} \chi_{6B_i} < \infty$  at each  $x$  but these sums are not necessarily uniformly bounded on  $X$ .

*Proof:* By Lemma 2.2 we can pick a disjoint subfamily  $\{B_i : B_i = B(x_{B_i}, r_{B_i})\}_{i \in E}$  with  $E \subset F$  satisfying (i). Moreover, we can assume that for  $i, j \in E$ , neither  $6B_i \subset 6B_j$  nor  $6B_j \subset 6B_i$ .

To prove (ii), we assume in contradiction that there exists some  $x \in X$  such that there exists an infinite family of balls  $\{B_i : i \in I_x \subset E\}$  such that  $x \in B_i$  for all  $i \in I_x$ . We will show that  $\liminf_{i \in I_x} r_{B_i} > 0$ . Otherwise, for any  $\epsilon > 0$  there exists  $i_\epsilon \in I_x$  such that  $r_{B_{i_\epsilon}} < \epsilon$ . Therefore, if  $B_0$  is any ball in the family  $\{B_i : i \in I_x\}$ , there exists  $r > 0$  such that  $B(x, r) \subset 6B_0$ . For  $\epsilon = \frac{r}{30}$ , we have  $x \in 6B_{i_\epsilon}$  and  $r_{6B_{i_\epsilon}} < \frac{r}{4}$ . This implies  $6B_{i_\epsilon} \subset 6B_0$  which is a contradiction.

Thus  $\liminf_{i \in I_x} r_{B_i} > 0$ . This together with the uniform boundedness of diameter of the family of balls shows that there exist  $m$  and  $M > 0$  such that  $m < r_{B_i} < M$  for all  $i \in I_x$ . Obviously,  $\cup_{i \in I_x} B(x_{B_i}, m) \subset B(x, 2M)$  and the balls  $\{B(x_{B_i}, m) : i \in I_x\}$  are pairwise disjoint. By Lemma 2.1, there exists a finite family of balls with radius  $\frac{m}{30}$  such that  $B(x, 2M) \subset \cup_{i=1}^K B(x_i, \frac{m}{30})$ . Therefore, there exist a ball, says  $B_k \in \{B(x_i, \frac{m}{30}) : i \in 1, \dots, K\}$ , and at least two balls  $B_1$  and  $B_2$  in  $\{B_i : i \in I_x\}$  such that  $B_k \cap \frac{1}{6}B_1 \not\subset \emptyset$  and  $B_k \cap \frac{1}{6}B_2 \not\subset \emptyset$ . Since  $\min\{r_{\frac{1}{6}B_1}, r_{\frac{1}{6}B_2}\} > \frac{1}{6}m = 5r_{B_k}$ , we have  $B_k \subset B_1 \cap B_2$ . This is a contradiction, because the family of balls  $\{B(x_{B_i}, m) : i \in I_x\}$  is pairwise disjoint. Our proof is completed.

We now give a Calderón-Zygmund decomposition on a non-homogenous space  $(X, \mu)$  which is an extension of a Calderón-Zygmund decomposition on the non-homogeneous space  $(\mathbb{R}^n, \mu)$  in [T1].

**Theorem 6.3** (*Calderón-Zygmund decomposition*) Assume  $1 \leq p < \infty$ . For any  $f \in L^p(\mu)$  and any  $\lambda > 0$  (with  $\lambda > \beta_0 \|f\|_p / \|\mu\|$  if  $\|\mu\| < \infty$ ), the following statements hold.

(a) *There exists a family of finite overlapping balls  $\{6Q_i\}_i$  such that  $\{Q_i\}_i$  is a pairwise disjoint family and*

$$\frac{1}{\mu(6^2Q_i)} \int_{Q_i} |f|^p d\mu > \frac{\lambda^p}{\beta_0}, \quad (22)$$

$$\frac{1}{\mu(6^2\eta Q_i)} \int_{\eta Q_i} |f|^p d\mu \leq \frac{\lambda^p}{\beta_0}, \text{ for all } \eta > 1, \quad (23)$$

$$|f| \leq \lambda \text{ a.e. } (\mu) \text{ on } X \setminus \bigcup_i 6Q_i. \quad (24)$$

(b) *For each  $i$ , let  $R_i$  be a  $(3 \times 6^2, C_{\lambda}^{\log_2 3 \times 6^2 + 1})$ -doubling ball concentric with  $Q_i$ , with  $l(R_i) > 6^2 l(Q_i)$  and denote  $\omega_i = \frac{\chi_{6Q_i}}{\sum_k \chi_{6Q_k}}$ . Then there exists a family of functions  $\varphi_i$  with constant signs and  $\text{supp } (\varphi_i) \subset R_i$  satisfying*

$$\int \varphi_i d\mu = \int_{6Q_i} f \omega_i d\mu, \quad (25)$$

$$\sum_i |\varphi_i| \leq \kappa \lambda, \quad (26)$$

(where  $\kappa$  is some constant which depends only on  $(X, \mu)$ ), and

(i)

$$\|\varphi_i\|_{\infty} \mu(R_i) \leq C \int_X |w_i f| d\mu \text{ if } p = 1; \quad (27)$$

(ii)

$$\|\varphi_i\|_{L^p(\mu)} \mu(R_i)^{1/p'} \leq \frac{C}{\lambda^{p-1}} \int_X |w_i f|^p d\mu \text{ if } 1 < p < \infty. \quad (28)$$

(c) For  $1 < p < \infty$ , if  $R_i$  is the smallest  $(3 \times 6^2, C_\lambda^{\log_2 3 \times 6^2 + 1})$ -doubling ball of the family  $\{3 \times 6^2 Q_i\}_{k \geq 1}$ , then

$$\|b\|_{H_{at}^{1,p}(\mu)} \leq \frac{C}{\lambda^{p-1}} \|f\|_{L^p(\mu)}^p \quad (29)$$

where  $b = \sum_i (w_i f - \varphi_i)$ .

*Proof:* For the sake of simplicity, we only give the proof for the case  $p = 1$  for (a) and (b). When  $p > 1$ , by setting  $g = f^p \in L^1(\mu)$ , we can reduce to the problem  $p = 1$ . Then, with a simple modification, we will obtain (28) instead of (27).

(a) Set  $E := \{x : |f(x)| > \lambda\}$ . For each  $x \in E$ , there exists some ball  $Q_x$  such that

$$\frac{1}{\mu(6^2 Q_x)} \int_{Q_x} |f| d\mu > \frac{\lambda}{\beta_0} \quad (30)$$

and such that if  $Q'_x$  is centered at  $x$  with  $l(Q'_x) > l(Q_x)$ , then

$$\frac{1}{\mu(6^2 Q'_x)} \int_{Q'_x} |f| d\mu \leq \frac{\lambda}{\beta_0}$$

Now we can apply Lemma 6.2 to get a family of balls  $\{Q_i\}_i \subset \{Q_x\}_x$  such that  $\sum_j \chi_{6Q_j}(x) < \infty$  for all  $x \in X$  and (22), (23) and (24) are satisfied.

(b) Assume first that the family of balls  $\{Q_i\}$  is finite. Without loss of generality, suppose that  $l(R_i) \leq l(R_{i+1})$ . The functions  $\varphi$  will be constructed of the form  $\varphi_i = \alpha_i \chi_{A_i}$ ,  $A_i \subset R_i$ .

First, set  $A_1 = R_1$  and  $\varphi_1 = \alpha_1 \chi_{R_1}$  such that  $\int \varphi_1 = \int_{6Q_1} f \omega_1$ . Assume that  $\varphi_1, \dots, \varphi_{k-1}$  have been constructed satisfying (25) and

$$\sum_{i=1}^{K-1} \varphi_i \leq \kappa \lambda,$$

where  $\kappa$  is some constant which will be fixed later. There are two cases:

**Case 1:** There exists some  $i \in \{1, \dots, k-1\}$  such that  $R_i \cap R_k \neq \emptyset$ . Let  $R_{s_1}, \dots, R_{s_m}$  be the family of  $R_1, \dots, R_{k-1}$  such that  $R_{s_j} \cap R_k \neq \emptyset$ . Since  $l(R_{s_j}) \leq l(R_k)$ ,  $R_{s_j} \subset 3R_k$ . By using  $R_k$  is  $(3 \times 6^2, C_\lambda^{\log_2 3 \times 6^2 + 1})$ -doubling and (23), we get

$$\begin{aligned} \sum_j |\varphi_{s_j}| &\leq \sum_j \int_X |f \omega_{s_j}| d\mu \\ &\leq C \sum_j \int_X \omega_{s_j} |f| d\mu \leq C \sum_j \int_{3R_k} |f| d\mu \leq C \lambda \mu(3 \cdot 6^2 R_k) \leq C_1 \lambda \mu(R_k). \end{aligned}$$

Therefore,

$$\mu\left\{\sum_j \|\varphi_{s_j}\| > 2C_1 \lambda\right\} \leq \frac{\mu(R_k)}{2}.$$

Thus,

$$\mu(A_k) \geq \frac{\mu(R_k)}{2}, A_k = R_k \cap \left\{\sum_j |\varphi_{s_j}| \leq 2C_1 \lambda\right\}.$$

The constant  $\alpha_k$  will be chosen such that  $\int \varphi_k = \int_{Q_k} f \omega_k d\mu$  where  $\varphi_k = \alpha_k \chi_{A_k}$ . Then we obtain

$$\alpha_k \leq \frac{C}{\mu(A_k)} \int_X w_i |f| d\mu \leq C \frac{2}{\mu(R_k)} \int_{\frac{1}{6^2} R_k} |f| d\mu \leq C_2 \lambda \quad (\text{by using (23)}).$$

If we choose  $\kappa = 2C_1 + C_2$ , (26) follows.

**Case 2:**  $R_i \cap R_k = \emptyset$  for all  $i = 1, \dots, k-1$ . Set  $A_k = R_k$  and  $\varphi_k = \alpha_k \chi_{R_k}$  such that  $\int \varphi_k = \int_{Q_k} f \omega_k d\mu$ . We also get (26).

By the construction of the functions  $\varphi_i$ , it is easy to see that  $\mu(R_i) \leq 2\mu(A_k)$ . Hence,

$$\|\varphi_i\|_\infty \mu(R_i) \leq C \alpha_i \mu(A_k) \leq C \int_X |f \omega_i| d\mu.$$

When the collection of balls  $\{Q_i\}$  is not finite, we can argue as in [T1, p.134]. This completes the proofs of (a) and (b).

(c) Since  $R_i$  is the smallest  $(3 \times 6^2, C_\lambda^{\log_2 3 \times 6^2 + 1})$ -doubling ball of the family  $\{3 \times 6^2 Q_i\}_{i \geq 1}$ , one has  $K_{Q_i, R_i} \leq C$ . For each  $i$ , we consider the atomic block  $b_i = f w_i - \varphi_i$  supported in ball  $R_i$ . By (22) and (28) we have

$$|b_i|_{H_{at}^{1,p}(\mu)} \leq \frac{C}{\lambda^{p-1}} \int_X |f w_i|^p d\mu$$

which implies

$$|b|_{H_{at}^{1,p}(\mu)} \leq \frac{C}{\lambda^{p-1}} \int_X \sum_i |f w_i|^p d\mu \leq \frac{C}{\lambda^{p-1}} \int_X \left(\sum_i w_i\right)^p |f|^p d\mu = \frac{C}{\lambda^{p-1}} \int_X |f|^p d\mu.$$

Our proof is completed.

Using the Calderón-Zygmund decomposition and a standard argument, see for example [J, pp.43-44] (also [T1, p.135]), we obtain the following interpolation result for a linear operator. For clarity and completeness, we sketch the proof below.

**Theorem 6.4** *Let  $T$  be a linear operator which is bounded from  $H_{at}^{1,\infty}(\mu)$  into  $L^1(\mu)$  and from  $L^\infty(\mu)$  into  $\text{RBMO}(\mu)$ . Then  $T$  can be extended to a bounded operator on  $L^p(\mu)$  for all  $1 < p < \infty$ .*

*Proof:* For simplicity we may assume that  $\|\mu\| = \infty$ . Let  $f$  be a function in  $L^\infty(\mu)$  with bounded support satisfying  $\int f d\mu = 0$ . Let us recall that the set of all such functions is dense in  $L^p(\mu)$  for all  $1 < p < \infty$ . For such functions  $f$ , we need only to show that

$$\|M^\sharp T f\|_{L^p(\mu)} \leq C \|f\|_{L^p(\mu)}, \quad 1 < p < \infty. \quad (31)$$

Once (31) is proved, Theorem 6.4 follows from Theorem 4.2.

For such a function  $f$  and  $\lambda > 0$ , we can decompose the function  $f$  as in Theorem 6.3

$$f := b + g = \sum_i (w_i f - \varphi_i) + g.$$

By (24) and (26), we have  $\|g\|_{L^\infty(\mu)} \leq C\lambda$ , and by (29)

$$\|b\|_{H_{at}^{1,p}(\mu)} \leq \frac{C}{\lambda^{p-1}} \|f\|_{L^p(\mu)}^p.$$

Since  $T$  is bounded from  $L^\infty(\mu)$  into  $\text{RBMO}(\mu)$ , we have

$$\|M^\sharp T g\|_{L^\infty(\mu)} \leq C_0 \lambda.$$

Therefore,

$$\{M^\sharp T f > (C_0 + 1)\lambda\} \subset \{M^\sharp T b > \lambda\}.$$

The fact that  $M^\sharp$  is of weak type  $(1, 1)$  gives

$$\mu\{M^\sharp T b > \lambda\} \leq C \frac{\|T b\|_{L^1(\mu)}}{\lambda}.$$

Moreover, since  $T$  is bounded from  $H_{at}^{1,\infty}(\mu)$  into  $L^1(\mu)$ , we have

$$\|T b\|_{L^1(\mu)} \leq C \|b\|_{H_{at}^{1,\infty}(\mu)} \leq \frac{C}{\lambda^{p-1}} \|f\|_{L^p(\mu)}^p.$$

This implies

$$\mu\{M^\sharp T f > (C_0 + 1)\lambda\} \leq C \frac{\|f\|_{L^p(\mu)}^p}{\lambda^p}.$$

So the sublinear operator  $M^\sharp T$  is of weak type  $(p, p)$  for all  $1 < p < \infty$ . By Marcinkiewicz interpolation theorem the operator  $M^\sharp T$  is bounded for all  $1 < p < \infty$ . This completes our proof.

## 6.2 The weak $(1, 1)$ boundedness of Calderón-Zygmund operators

**Theorem 6.5** *If a Calderón-Zygmund operator  $T$  is bounded on  $L^2(\mu)$ , then  $T$  is of weak type  $(1, 1)$ .*

*Proof:* Let  $f \in L^1(\mu)$  and  $\lambda > 0$ . We can assume that  $\lambda > \beta_0 \|f\|_{L^1(\mu)} / \|\mu\|$ . Otherwise, there is nothing to prove. Using the same notations as in Theorem 6.3 with  $R_i$  which is chosen as the smallest  $(3 \times 6^2, C_\lambda^{\log_2 3 \times 6^2 + 1})$ -doubling ball of the family  $\{3 \times 6^2 Q_i\}_{k \geq 1}$ , we can write  $f = g + b$ , with

$$g = f \chi_{X \setminus \cup_i 6Q_i} + \sum_i \varphi_i$$

and

$$b := \sum_i b_i = \sum_i (w_i f - \varphi_i).$$

Taking into account (22), one has

$$\mu(\cup_i 6^2 Q_i) \leq \frac{C}{\lambda} \int_{Q_i} |f| d\mu \leq \frac{C}{\lambda} \int_X |f| d\mu$$

where in the last inequality we use the pairwise disjoint property of family  $\{Q_i\}_i$ . We need only to show that

$$\mu\{x \in X \setminus \cup_i 6^2 Q_i : |Tf(x)| > \lambda\} \leq \frac{C}{\lambda} \int_X |f| d\mu.$$

We have

$$\begin{aligned} \mu\{x \in X \setminus \cup_i 6^2 Q_i : |Tf(x)| > \lambda\} &\leq \mu\{x \in X \setminus \cup_i 6^2 Q_i : |Tg(x)| > \lambda/2\} \\ &\quad + \mu\{x \in X \setminus \cup_i 6^2 Q_i : |Tb(x)| > \lambda/2\} := I_1 + I_2. \end{aligned}$$

Let us estimate the term  $I_1$  related to the “good part” first. Since  $|g| \leq C\lambda$  then

$$\mu\{x \in X \setminus \cup_i 6^2 Q_i : |Tg(x)| > \lambda/2\} \leq \frac{C}{\lambda^2} \int |g|^2 d\mu \leq \frac{C}{\lambda} \int |g| d\mu.$$

Furthermore, we have

$$\begin{aligned} \int |g| d\mu &\leq \int_{X \setminus \cup_i 6Q_i} |f| d\mu + \sum_i \int_{R_i} |\varphi_i| \\ &\leq \int_X |f| d\mu + \sum_i \mu(R_i) \|\varphi_i\|_{L^\infty(\mu)} \\ &\leq \int_X |f| d\mu + C \sum_i \int_X |fw_i| d\mu \\ &\leq C \int_X |f| d\mu. \end{aligned}$$



Therefore,

$$\mu\{x \in X \setminus \cup_i 6^2 Q_i : |Tg(x)| > \lambda/2\} \leq \frac{C}{\lambda} \int |f| d\mu.$$

For the term  $I_2$ , we have

$$\begin{aligned} I_2 &\leq \frac{C}{\lambda} \sum_i \left( \int_{X \setminus 2R_i} |Tb_i| d\mu + \int_{2R_i} |T\varphi_i| d\mu + \int_{2R_i \setminus 6^2 Q_i} |Tw_i f| d\mu \right) \\ &\leq \frac{C}{\lambda} \sum_i \left( K_{i1} + K_{i2} + K_{i3} \right) \end{aligned}$$

Note that  $\int b_i d\mu = 0$  for all  $i$ . We have, by (3),

$$\begin{aligned} K_{i1} &= \int_{X \setminus 2R_i} |Tb_i| d\mu \leq C \int |b_i| d\mu \\ &\leq \int_X |fw_i| d\mu + \int_{R_i} |\varphi_i| d\mu \\ &\leq \int_X |fw_i| d\mu + \mu(R_i) \|\varphi_i\|_{L^\infty(\mu)} \\ &\leq C \sum_i \int_X |fw_i| d\mu \\ &\leq C \sum_i \int_X |f| d\mu. \end{aligned}$$

On the other hand, by the  $L^2$  boundedness of  $T$  and  $R_i$  is a  $(3 \times 6^2, C_\lambda^{\log_2 3 \times 6^2 + 1})$ -doubling ball, we get

$$\begin{aligned} K_{i2} &\leq \left( \int_{2R_i} |T\varphi_i|^2 \right)^{1/2} (\mu(2R_i))^{1/2} \\ &\leq \left( \int_{2R_i} |\varphi_i|^2 \right)^{1/2} (\mu(2R_i))^{1/2} \\ &\leq C \|\varphi_i\|_{L^\infty(\mu)} \mu(2R_i) \\ &\leq C \int |w_i f| d\mu. \end{aligned}$$

Moreover, taking into account the fact that  $\text{supp } w_i f \subset 6Q_i$ , for  $x \in 2R_i \setminus 6^2 Q_i$  we have, by Lemma 6.1,

$$K_{i3} \leq C \int_{2R_i \setminus 6^2 Q_i} \frac{1}{\lambda(x_{Q_i}, d(x, x_{Q_i}))} d\mu(x) \times \int_X |w_i f| d\mu.$$

Hence we obtain

$$I_2 \leq \frac{C}{\lambda} \sum_i \int_X |w_i f| d\mu \leq \frac{C}{\lambda} \sum_i \int_X |f| d\mu$$

and the proof is completed.

### 6.3 Cotlar inequality

We note that from the weak type  $(1, 1)$  estimate of  $T$ , we can obtain a Cotlar inequality on  $T$ . More precisely, we have the following result.

**Theorem 6.6** *Assume that  $T$  is a Calderón-Zygmund operator and that  $T$  is bounded on  $L^2(X, \mu)$ . Then there exists a constant  $C > 0$  such that for any bounded function  $f$  with compact support and  $x \in X$  we have*

$$T_*f(x) \leq C \left( M_{6,\eta}(Tf)(x) + M_{(5)}f(x) \right)$$

where

$$M_{p,\rho}f(x) = \sup_{Q \ni x} \left( \frac{1}{\mu(\rho Q)} \int_Q |f|^p \right).$$

*Proof:* For any  $\epsilon > 0$  and  $x \in X$ , let  $Q_x$  be the biggest  $(6, \beta)$ -doubling ball centered  $x$  of the form  $6^{-k}\epsilon$ ,  $k \geq 1$  and  $\beta > 6^n$ . Assume that  $Q_x = B(x, 6^{-k_0}\epsilon)$ . Then, we can break  $f = f_1 + f_2$ , where  $f_1 = f\chi_{\frac{6}{5}Q_x}$ . Obviously  $T_*f_1(x) = 0$ . This follows that

$$T_*f(x) \leq |Tf_2(x)| + \left| \int_{d(x,y) \leq \epsilon} K(x,y)f_2(y)d\mu(y) \right| = I_1 + I_2.$$

Let us estimate  $I_1$  first. For any  $z \in Q_x$ , we have

$$|Tf_2(x)| \leq |Tf_2(x) - Tf_2(z)| + |Tf(z)| + |Tf_1(z)|. \quad (32)$$

On the other hand, it follows from (3) that

$$\begin{aligned} |Tf_2(x) - Tf_2(z)| &\leq \int_{X \setminus \frac{6}{5}Q_x} |K(x,y) - K(z,y)| |f(y)| d\mu(y) \\ &\leq C \int_{X \setminus \frac{6}{5}Q_x} \frac{d(x,z)^\delta}{d(x,y)^\delta \lambda(x, d(x,y))} |f(y)| d\mu(y) \\ &\leq C \int_{X \setminus Q_x} \frac{r_{Q_x}^{n+\delta}}{d(x,y)^\delta \lambda(x, d(x,y))} |f(y)| d\mu(y) \\ &\leq C \sum_{k=0}^{\infty} \int_{6^{k+1}Q_x \setminus 6^kQ_x} \frac{r_{Q_x}^\delta}{d(x,y)^\delta \lambda(x, d(x,y))} |f(y)| d\mu(y) \\ &\leq C \sum_{k=0}^{\infty} \int_{6^{k+1}Q_x \setminus 6^kQ_x} \frac{r_{Q_x}^\delta}{(6^k r_{Q_x})^\delta \lambda(x, 6^k r_{Q_x})} |f(y)| d\mu(y) \\ &\leq C \sum_{k=0}^{\infty} 6^{-k} \frac{\mu(6 \times 6^{k+1}Q_x)}{\lambda(x, 6^k r_{Q_x})} \frac{1}{\mu(6 \times 6^{k+1}Q_x)} \int_{6^{k+1}Q_x} |f(y)| d\mu(y) \\ &\leq C \sum_{k=0}^{\infty} 6^{-k} M_{(6)}f(x) = CM_{(6)}f(x). \end{aligned}$$

This together with (33) implies

$$|Tf_2(x)| \leq CM_{(6)}f(x) + |Tf_1(z)| + |Tf(z)| \quad (33)$$

for all  $z \in Q_x$ .

At this stage, taking the  $L^\eta(Q_x, \frac{d\mu(x)}{\mu(Q)})$ -norm with respect to  $z$ , we have

$$|Tf_2(x)| \leq CM_{(6)}f(x) + \left( \frac{1}{\mu(Q_x)} \int_{Q_x} |Tf_1(z)|^\eta d\mu(z) \right)^{1/\eta} + \left( \frac{1}{\mu(Q_x)} \int_{Q_x} |Tf(z)|^\eta d\mu(z) \right)^{1/\eta}.$$

By the Kolmogorov inequality and the weak type  $(1, 1)$  boundedness of  $T$ , we have, for  $\eta < 1$ ,

$$\begin{aligned} \left( \frac{1}{\mu(Q_x)} \int_{Q_x} |Tf_1(z)|^\eta d\mu(z) \right)^{1/\eta} &\leq \frac{1}{\mu(Q_x)} \int_{\frac{6}{5}Q_x} |f_1(z)| d\mu(z) \\ &\leq \frac{C}{\mu(15Q_x)} \int_{\frac{6}{5}Q_x} |f_1(z)| d\mu(z) \quad (\text{since } Q_x \text{ is } (6, \beta)\text{-doubling}) \\ &\leq CM_{(5)}f(x). \end{aligned}$$

Furthermore, since  $Q_x$  is  $(6, \beta)$ -doubling,

$$\left( \frac{1}{\mu(Q_x)} \int_{Q_x} |Tf(z)|^\eta d\mu(z) \right)^{1/\eta} \leq CM_{\eta,6}M(Tf)(x).$$

Therefore,  $I_1 \leq CM_{(6)}f(x) + CM_{\eta,6}M(Tf)(x)$ .

For the term  $I_2$  we have

$$\begin{aligned} I_2 &\leq \int_{d(x,y) \leq \epsilon} |K(x,y)| |f_2(y)| d\mu(y) \\ &\leq C \int_{B(x,\epsilon) \setminus B(x,6^{-k_0}\epsilon)} \frac{1}{\lambda(x, d(x,y))} |f_2(y)| d\mu(y) \\ &\leq C \sum_{k=0}^{k_0-1} \int_{B(x,6^{k+1-k_0}\epsilon) \setminus B(x,6^{k-k_0}\epsilon)} \frac{1}{\lambda(x, d(x,y))} |f_2(y)| d\mu(y) \\ &\leq C \sum_{k=0}^{k_0-1} \int_{B(x,6^{k+1-k_0}\epsilon)} \frac{1}{\lambda(x, 6^{k-k_0}\epsilon)} |f_2(y)| d\mu(y) \\ &\leq C \sum_{k=0}^{k_0-1} \frac{\mu(x, 6 \times 6^{k+1-k_0}\epsilon)}{\lambda(x, 6 \times 6^{k+1-k_0}\epsilon)} \frac{1}{\mu(x, 6 \times 6^{k+1-k_0}\epsilon)} \int_{B(x,6^{k+1-k_0}\epsilon)} |f_2(y)| d\mu(y) \\ &\leq C \sum_{k=0}^{k_0-1} \frac{\mu(x, 6 \times 6^{k+1-k_0}\epsilon)}{\lambda(x, 6 \times 6^{k+1-k_0}\epsilon)} M_{(6)}f(x). \end{aligned}$$

At this stage, by repeating the argument in the proof of Lemma 6.1, we have

$$\sum_{k=0}^{k_0-1} \frac{\mu(x, 6 \times 6^{k+1-k_0}\epsilon)}{\lambda(x, 6 \times 6^{k+1-k_0}\epsilon)} \leq C.$$

Therefore,

$$I_2 \leq CM_{(6)}f(x).$$

This completes our proof.

**Remark 6.7** From the boundedness of  $M_{6,\eta}(\cdot)$  and  $M_{(5)}(\cdot)$ , the Cotlar inequality tells us that if  $T$  is bounded on  $L^2(X, \mu)$  then the maximal operator  $T_*$  is bounded on  $L^p(X, \mu)$  for  $1 < p < \infty$ . Note that the endpoint estimate of  $T_*$  will be investigated in [AD].

The Calderón-Zygmund decomposition Theorem 6.3 does not require the property (v) of  $\lambda(\cdot, \cdot)$ .

## 7 The boundedness of Calderón-Zygmund operators

The main results of this section are Theorems 7.1, 7.3 and 7.6.

### 7.1 The boundedness of Calderón-Zygmund operators from $L^\infty$ to RBMO space

The following result shows that on a non-homogeneous space  $(X, \mu)$ , a Calderón-Zygmund operator which is bounded on  $L^2$  is also bounded from  $L^\infty(\mu)$  into the regularized BMO space RBMO( $\mu$ ).

**Theorem 7.1** Assume that  $T$  is a Calderón-Zygmund operator and  $T$  is bounded on  $L^2(\mu)$ , then  $T$  is bounded from  $L^\infty(\mu)$  into RBMO( $\mu$ ). Therefore, by interpolation and duality,  $T$  is bounded on  $L^p(\mu)$  for all  $1 < p < \infty$ .

*Proof:* We use the RBMO characterizations (9) and (10). The condition (9) can be obtained by the standard method used in the case of doubling measure. We omit the details here.

We will check (10). To do this, we have to show that

$$|m_Q(Tf) - m_R(Tf)| \leq CK_{Q,R} \left( \frac{\mu(6Q)}{\mu(Q)} + \frac{\mu(6R)}{\mu(R)} \right) \|f\|_{L^\infty(\mu)}$$

for all  $Q \subset R$ .

Let  $N$  be the first integer  $k$  such that  $R \subset 6^k Q$ . We denote  $Q_R = 6^{N+1}Q$ . Then for  $x \in Q$  and  $y \in R$ , we set

$$\begin{aligned} Tf(x) - Tf(y) &= Tf\chi_{6Q}(x) + Tf\chi_{6^N Q \setminus 6Q}(x) + Tf\chi_{X \setminus Q_R}(x) \\ &\quad - (Tf\chi_{Q_R}(y) + Tf\chi_{X \setminus Q_R}(y)) \\ &\leq |Tf\chi_{6Q}(x)| + |Tf\chi_{6^N Q \setminus 6Q}(x)| \\ &\quad + |Tf\chi_{X \setminus Q_R}(x) - Tf\chi_{X \setminus Q_R}(y)| + |Tf\chi_{Q_R}(y)| \\ &\leq I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Let us estimate  $I_3$  first. We have

$$\begin{aligned}
I_3 &\leq \int_{X \setminus Q_R} |K(x, z) - K(y, z)| |f(z)| d\mu(z) \\
&\leq \sum_{k=N+1}^{\infty} \int_{6^{k+1}Q \setminus 6^kQ} \frac{d(x, y)^\delta}{d(x, z)^\delta \lambda(x, d(x, z))} |f(z)| d\mu(z) \\
&\leq \sum_{k=N+1}^{\infty} 6^{-(k-N)\delta} \frac{\mu(6^{k+1}Q)}{\lambda(x, 6^{k+1}r_Q)} \|f\|_{L^\infty(\mu)} \\
&\leq C \sum_{k=N+1}^{\infty} 6^{-(k-N)\delta} \frac{\mu(6^{k+1}Q)}{\lambda(x, 6^{k+1}r_Q)} \|f\|_{L^\infty(\mu)} \\
&\leq C \sum_{k=N+1}^{\infty} 6^{-(k-N)\delta} \|f\|_{L^\infty(\mu)} = C \|f\|_{L^\infty(\mu)},
\end{aligned}$$

where in the last inequality we use the fact that  $\mu(6^{k+1}Q) \leq \lambda(x, 6^{k+1}r_Q)$ , since  $x \in Q \subset 2^{k+1}Q$ .

As to the term  $I_2$ , we have

$$\begin{aligned}
Tf \chi_{6^N Q \setminus 6Q}(x) &\leq \int_{6^N Q \setminus 6Q} |K(x, y)| |f(y)| d\mu(y) \\
&\leq \int_{6^N Q \setminus 6Q} \frac{C}{\lambda(x_Q, d(x_Q, y))} |f(y)| d\mu(y) \\
&\leq K_{6Q, 6^N Q} \|f\|_{L^\infty(\mu)}.
\end{aligned} \tag{34}$$

Therefore,  $I_2 \leq CK_{Q,R} \|f\|_\infty$ . So, we get

$$Tf(x) - Tf(y) = Tf \chi_{6Q}(x) + CK_{Q,R} \|f\|_{L^\infty(\mu)} + |Tf \chi_{Q_R}(y)| + C \|f\|_{L^\infty(\mu)}.$$

Taking the mean over  $Q$  and  $R$  for  $x$  and  $y$ , respectively, we have

$$\begin{aligned}
|m_Q(Tf) - m_R(Tf)| &\leq m_Q(|Tf \chi_{6Q}|) + CK_{Q,R} \|f\|_{L^\infty(\mu)} + |Tf \chi_{Q_R}(y)| \\
&\quad + C \|f\|_{L^\infty(\mu)} + m_R(Tf \chi_{Q_R}).
\end{aligned}$$

For the boundedness on  $L^2(\mu)$  of  $T$ , we have

$$\begin{aligned}
m_Q(|Tf \chi_{6Q}|) &\leq \left( \frac{1}{\mu(Q)} \int_Q |Tf \chi_{6Q}|^2 \right)^{1/2} \\
&\leq C \left( \frac{\mu(6Q)}{\mu(Q)} \right)^{1/2} \|f\|_{L^\infty(\mu)} \\
&\leq C \left( \frac{\mu(6Q)}{\mu(Q)} \right) \|f\|_{L^\infty(\mu)}.
\end{aligned}$$

Next, we write

$$m_R(Tf \chi_{Q_R}) \leq m_R(Tf \chi_{Q_R \cap 6R}) + m_R(Tf \chi_{Q_R \setminus 6R}).$$

By similar argument in estimate of  $m_Q(|Tf\chi_{6Q}|)$ , the term  $m_R(Tf\chi_{Q_R\cap 6R})$  is dominated by

$$C\left(\frac{\mu(6R)}{\mu(R)}\right)\|f\|_{L^\infty(\mu)}.$$

The second term  $m_R(Tf\chi_{Q_R\setminus 6R})$  can be treated as in (34). Since  $r_{Q_R} \approx r_R$ , we have

$$m_R(Tf\chi_{Q_R\setminus 6R}) \leq C\|f\|_{L^\infty(\mu)}$$

To sum up, we have

$$\begin{aligned} |m_Q(Tf) - m_R(Tf)| &\leq CK_{Q,R}\|f\|_{L^\infty(\mu)} + \left(\frac{\mu(6Q)}{\mu(Q)} + \frac{\mu(6R)}{\mu(R)}\right)\|f\|_{L^\infty(\mu)} \\ &\leq CK_{Q,R}\left(\frac{\mu(6Q)}{\mu(Q)} + \frac{\mu(6R)}{\mu(R)}\right)\|f\|_{L^\infty(\mu)}. \end{aligned}$$

**Remark 7.2** *By similar argument in [T1, Theorem 2.11], we can replace the assumption of  $L^2(\mu)$  boundedness by the weaker assumption: for any ball  $B$  and any function  $a$  supported on  $B$ ,*

$$\int_B |Ta| d\mu \leq C\|a\|_{L^\infty(\mu)} \mu(6B)$$

*uniformly on  $\epsilon > 0$ .*

## 7.2 The boundedness of Calderón-Zygmund operators on Hardy spaces

We now show that an  $L^2$  bounded Calderón-Zygmund operator maps the atomic Hardy space boundedly into  $L^1$ .

**Theorem 7.3** *Assume that  $T$  is a Calderón-Zygmund operator and  $T$  is bounded on  $L^2(X, \mu)$ , then  $T$  is bounded from  $H_{at}^{1,\infty}(\mu)$  into  $L^1(X, \mu)$ . Therefore, by interpolation and duality,  $T$  is bounded on  $L^p(\mu)$  for all  $1 < p < \infty$ .*

*Proof:* By [HoM, Lemm 4.1], it is enough to show that

$$\|Tb\|_{L^1(\mu)} \leq C\|b\|_{H_{at}^{1,\infty}(\mu)} \tag{35}$$

for any atomic block  $b$  with  $\text{supp } b \subset B$  and  $b = \sum_j \lambda_j a_j$  where the  $a_j$ 's are functions satisfying (a) and (b) in definition of atomic blocks. At this stage we can use the same argument as in [T1, Theorem 4.2] with minor modifications as in Theorem 7.1 to obtain the estimate (35). We omit the details here.

### 7.3 Commutators of Calderón-Zygmund operators with RBMO functions

In this section we assume that the dominating function  $\lambda$  satisfies  $\lambda(x, ar) = a^m \lambda(x, r)$  for all  $x \in X$  and  $a, r > 0$ . Then, for two balls  $B, Q$  such that  $B \subset Q$  we can define the coefficient  $K'_{B,Q}$  as follows: let  $N_{B,Q}$  be the smallest integer satisfying  $6^{N_{B,Q}} r_B \geq r_Q$ , we set

$$K'_{B,Q} := 1 + \sum_{k=1}^{N_{B,Q}} \frac{\mu(6^k B)}{\lambda(x_B, 6^k r_B)}. \quad (36)$$

It is not difficult to show that the coefficient  $K_{B,Q} \approx K'_{B,Q}$ . Note that in the definition of  $K'_{B,Q}$  we can replace 6 by any number  $\eta > 1$ .

To establish the boundedness of commutators of Calderón-Zygmund operators with RBMO functions, we need the following two lemmas. Note that these lemmas are similar to those in [T1]. However, due to the difference of choices of coefficient  $K_{Q,R}$ , we would like to provide the proof for the first one. Meanwhile, the proof of Lemma 7.5 is completely analogous to that of Lemma 9.3 in [T1], hence we omit the details.

**Lemma 7.4** *If  $B_i = B(x_0, r_i), i = 1, \dots, m$  are concentric balls  $B_1 \subset B_2 \subset \dots \subset B_m$  with  $K_{B_i, B_{i+1}} > 2$  for  $i = 1, \dots, m-1$  then*

$$\sum_{i=1}^{m-1} K_{B_i, B_{i+1}} \leq 2K_{B_1, B_m}. \quad (37)$$

*Proof:* By definition,

$$K_{B_i, B_{i+1}} = 1 + \int_{r_i \leq d(x, x_0) \leq r_{i+1}} \frac{1}{\lambda(x_0, d(x, x_0))} d\mu(x).$$

Since  $K_{B_i, B_{i+1}} > 2$ , we have

$$K_{B_i, B_{i+1}} < 2 \int_{r_i \leq d(x, x_0) \leq r_{i+1}} \frac{1}{\lambda(x_0, d(x, x_0))} d\mu(x)$$

for all  $i = 1, \dots, m-1$ .

This implies

$$\begin{aligned} \sum_{i=1}^{m-1} K_{B_i, B_{i+1}} &< 2 \sum_{i=1}^{m-1} \int_{r_i \leq d(x, x_0) \leq r_{i+1}} \frac{1}{\lambda(x_0, d(x, x_0))} d\mu(x) \\ &\leq 2 \int_{r_1 \leq d(x, x_0) \leq r_m} \frac{1}{\lambda(x_0, d(x, x_0))} d\mu(x) \\ &\leq 2K_{B_1, B_m}. \end{aligned}$$

**Lemma 7.5** *There exists some constant  $P_0$  such that if  $x \in X$  is some fixed point and  $\{f_B\}_{B \ni x}$  is collection of numbers such that  $|f_Q - f_R| \leq C_x$  for all doubling balls  $Q \subset R$  with  $x \in Q$  and  $K_{Q,R} \leq P_0$ , then*

$$|f_Q - f_R| \leq CK_{Q,R}C_x \text{ for all doubling balls } Q \subset R \text{ with } x \in Q.$$

**Theorem 7.6** *If  $b \in \text{RBMO}(\mu)$  and  $T$  is a Calderón-Zygmund bounded on  $L^2(\mu)$ , then the commutator  $[b, T]$  defined by*

$$[b, T](f) = bT(f) - T(bf)$$

*is bounded on  $L^p(\mu)$  for  $1 < p < \infty$ .*

*Proof:* For  $1 < p < \infty$  we will show that

$$M^\sharp([b, T]f)(x) \leq C\|b\|_{\text{RBMO}(\mu)} \left( M_{p,5}f(x) + M_{p,6}Tf(x) + T_*f(x) \right), \quad (38)$$

where

$$M_{p,\rho}f(x) = \sup_{Q \ni x} \left( \frac{1}{\mu(\rho Q)} \int_Q |f|^p \right).$$

Once (38) is proved, it follows from the boundedness of  $T_*$  on  $L^p(\mu)$  and  $M_{p,\rho}$  on  $L^r(\mu)$ ,  $r > p$  and  $\rho \geq 5$ , and from a standard argument that we can obtain the boundedness of  $[b, T]$  on  $L^p(\mu)$ .

Let  $\{b_B\}$  be a family of numbers satisfying

$$\int_B |b - b_B| d\mu \leq 2\mu(6B)\|b\|_{\text{RBMO}}$$

for balls  $B$ , and

$$|b_Q - b_R| \leq 2K_{Q,R}\|b\|_{\text{RBMO}}$$

for balls  $Q \subset R$ . Denote

$$h_Q := m_Q(T((b - b_Q)f\chi_{X \setminus \frac{6}{5}Q})).$$

We will show that

$$\frac{1}{\mu(6B)} \int_B |[b, T]f - h_Q| d\mu \leq C\|b\|_{\text{RBMO}} (M_{p,5}f(x) + M_{p,6}Tf(x)) \quad (39)$$

for all  $x$  and  $B$  with  $x \in B$ , and

$$|h_Q - h_R| \leq C\|b\|_{\text{RBMO}} (M_{p,5}f(x) + T_*f(x)) K_{Q,R}^2 \quad (40)$$

for all  $x \in Q \subset R$ .

The proof of (39) is similar to that in Theorem 9.1 in [T1] with minor modifications and we omit it here.

It remains to check (40). For two balls  $Q \subset R$ , let  $N$  be an integer such that  $(N - 1)$  is



the smallest number satisfying  $r_R \leq 6^{N-1}r_Q$ . Then, we break the term  $|h_Q - h_R|$  into five terms:

$$\begin{aligned}
& |m_Q(T((b - b_Q)f\chi_{X \setminus \frac{6}{5}Q}) - m_R(T((b - b_R)f\chi_{X \setminus \frac{6}{5}R}))| \\
& \leq |m_Q(T((b - b_Q)f\chi_{6Q \setminus \frac{6}{5}Q}))| + |m_Q(T((b_Q - b_R)f\chi_{X \setminus 6Q}))| \\
& \quad + |m_Q(T((b - b_R)f\chi_{6^N Q \setminus 6Q}))| \\
& \quad + |m_Q(T((b - b_R)f\chi_{X \setminus 6^N Q}) - m_R(T((b - b_R)f\chi_{X \setminus 6^N Q}))| \\
& \quad + |m_R(T((b - b_R)f\chi_{6^N Q \setminus \frac{6}{5}R}))| \\
& = M_1 + M_2 + M_3 + M_4 + M_5.
\end{aligned}$$

Let us estimate  $M_1$  first. For  $y \in Q$  we have, by Proposition 3.2

$$\begin{aligned}
& |T((b - b_Q)f\chi_{6Q \setminus \frac{6}{5}Q})(x)| \\
& \leq \frac{C}{\lambda(x, r_Q)} \int_{6Q} |b - b_Q| |f| d\mu \\
& \leq \frac{\mu(30Q)}{\lambda(x, 30r_Q)} \left( \frac{1}{\mu(5 \times 6Q)} \int_{6Q} |b - b_Q|^{p'} d\mu \right)^{1/p'} \left( \frac{1}{\mu(5 \times 6Q)} \int_{6Q} |f|^p d\mu \right)^{1/p} \\
& C \|b\|_{\text{RBMO}} M_{p,5} f(x).
\end{aligned}$$

The term  $M_5$  can be treated by similar way. So, we have  $M_1 + M_5 \leq C \|b\|_{\text{RBMO}} M_{p,5} f(x)$ . For the term  $M_2$ , we have for  $x, y \in Q$

$$\begin{aligned}
|Tf\chi_{X \setminus 6Q}(y)| &= \left| \int_{X \setminus 6Q} K(y, z) f(z) d\mu(z) \right| \\
&\leq \left| \int_{X \setminus 6Q} K(x, z) f(z) d\mu(z) \right| + \int_{X \setminus 6Q} |K(y, z) - K(x, z)| |f(z)| d\mu(z) \\
&\leq T_* f(x) + C M_{p,5} f(x).
\end{aligned}$$

This implies

$$|m_Q(T((b_Q - b_R)f\chi_{X \setminus 6Q}))| \leq C K_{Q,R}(T_* f(x) + M_{p,6} f(x)).$$

For the term  $M_4$ , we have, for  $y, z \in R$

$$\begin{aligned}
& |T((b - b_R)f\chi_{X \setminus 6^N Q}(y) - T((b - b_R)f\chi_{X \setminus 6^N Q}(z))| \\
& \leq \int_{X \setminus 2R} |K(y, w) - K(z, w)| |(b(w) - b_R)| |f(x)| d\mu(w) \\
& \leq \int_{X \setminus 2R} \frac{d(y, z)^\delta}{d(w, y)^\delta \lambda(y, d(w, y))} |(b(w) - b_R)| |f(x)| d\mu(w) \\
& \leq \sum_{k=1}^{\infty} \int_{2^{k+1}R \setminus 2^k R} \frac{d(y, z)^\delta}{d(w, y)^\delta \lambda(y, d(w, y))} |(b(w) - b_R)| |f(x)| d\mu(w) \\
& \leq \sum_{k=1}^{\infty} 2^{-k\delta} \frac{1}{\lambda(y, 2^{k-1}r_R)} \int_{2^{k+1}R} |(b(w) - b_R)| |f(x)| d\mu(x)
\end{aligned}$$

By Hölder inequality we have

$$\begin{aligned}
& |T((b - b_R)f\chi_{X \setminus 6^N Q}(y) - T((b - b_R)f\chi_{X \setminus 6^N Q}(z))| \\
& \leq C \sum_{k=1}^{\infty} 2^{-k\delta} \frac{\mu(5 \times 2^{k+1}R)}{\lambda(y, 5 \times 2^{k+1}R)} \left( \frac{1}{\mu(5 \times 2^{k+1}R)} \int_{2^{k+1}R} |b - b_R|^{p'} d\mu \right)^{1/p'} \\
& \quad \left( \frac{1}{\mu(5 \times 2^{k+1}R)} \int_{2^{k+1}R} |f|^p d\mu \right)^{1/p} \\
& \leq C \sum_{k=1}^{\infty} 2^{-k\delta} \left[ \left( \frac{1}{\mu(5 \times 2^{k+1}R)} \int_{2^{k+1}R} |b - b_{2^{k+1}R}|^{p'} d\mu \right)^{1/p'} \right. \\
& \quad \left. + \left( \frac{1}{\mu(5 \times 2^{k+1}R)} \int_{2^{k+1}R} |b_R - b_{2^{k+1}R}|^{p'} d\mu \right)^{1/p'} \right] \left( \frac{1}{\mu(5 \times 2^{k+1}R)} \int_{2^{k+1}R} |f|^p d\mu \right)^{1/p} \\
& \leq C \sum_{k=1}^{\infty} C(k+1)2^{-k\delta} \|b\|_{\text{RBMO}} M_{p,5}f(x) \\
& \leq C \|b\|_{\text{RBMO}} M_{p,5}f(x).
\end{aligned}$$

Taking the mean over  $Q$  and  $R$  for  $y$  and  $z$  respectively, we obtain

$$M_4 \leq C \|b\|_{\text{RBMO}} M_{p,5}f(x).$$

Concerning the last estimate for  $M_3$ , we have for  $y \in Q$

$$\begin{aligned}
& |T((b - b_R)f\chi_{6^N Q \setminus 6Q}(y))| \\
& \leq C \sum_{k=1}^{N-1} \frac{1}{\lambda(6^k Q)} \int_{6^{k+1}Q \setminus 6^k Q} |b - b_R| |f| d\mu \\
& \leq C \sum_{k=1}^{N-1} \frac{1}{\lambda(y, 6^k Q)} \left[ \int_{6^{k+1}Q \setminus 6^k Q} |b - b_{5^{k+1}Q}| |f| d\mu + \int_{6^{k+1}Q \setminus 6^k Q} |b_R - b_{6^{k+1}Q}| |f| d\mu \right] \\
& \leq C \sum_{k=1}^{N-1} \frac{\mu(5 \times 6^{k+1}Q)}{\lambda(x_Q, 6^k Q)} \left[ \frac{1}{\mu(6^{k+2}Q)} \int_{6^{k+1}Q \setminus 6^k Q} |b - b_{6^{k+1}Q}| |f| d\mu \right. \\
& \quad \left. + \frac{1}{\mu(5 \times 6^{k+1}Q)} \int_{6^{k+1}Q \setminus 6^k Q} |b_R - b_{6^{k+1}Q}| |f| d\mu \right].
\end{aligned} \tag{41}$$

By Hölder inequality and a similar argument to the estimate of the term  $M_4$ , we have

$$\frac{1}{\mu(5 \times 6^{k+2}Q)} \int_{6^{k+1}Q \setminus 6^k Q} |b - b_{6^{k+1}Q}| |f| d\mu \leq \|b\|_{\text{RBMO}} M_{p,5}f(x)$$

and

$$\frac{1}{\mu(5 \times 6^{k+1}Q)} \int_{6^{k+1}Q \setminus 6^k Q} |b_R - b_{6^{k+1}Q}| |f| d\mu \leq CK_{Q,R} \|b\|_{\text{RBMO}} M_{p,5}f(x).$$

These two estimates together with (41) give

$$|T((b - b_R)f\chi_{6^N Q \setminus 6Q}(y))| \leq CK_{Q,R}^2 \|b\|_{\text{RBMO}} M_{p,5} f(x).$$

This implies  $M_3 \leq CK_{Q,R}^2 \|b\|_{\text{RBMO}} M_{p,5} f(x)$ . From the estimates  $M_1, M_2, M_3, M_4, M_5$ , we obtain (40).

To obtain (38) from (39) and (40), we use a trick of [T1]. From (39), if  $Q$  is a doubling ball and  $x \in Q$ , we have

$$\begin{aligned} |m_Q([b, T]f) - h_Q| &\leq \frac{1}{\mu(Q)} \int_Q |[b, T]f - h_Q| d\mu \\ &\leq C \|b\|_{\text{RBMO}} (M_{p,5} f(x) + M_{p,6} T f(x)). \end{aligned} \quad (42)$$

Also, for any ball  $Q \ni x$  (non doubling, in general),  $K_{Q,\tilde{Q}} \leq C$ , and then by (39) and (40) we have

$$\begin{aligned} &\frac{1}{\mu(6Q)} \int_Q |[b, T]f - m_{\tilde{Q}}[b, T]f| d\mu \\ &\leq \frac{1}{\mu(6Q)} \int_Q |[b, T]f - h_Q| d\mu + |h_Q - h_{\tilde{Q}}| + |h_{\tilde{Q}} - m_{\tilde{Q}}[b, T]f| \\ &\leq C \|b\|_{\text{RBMO}(\mu)} (M_{p,5} f(x) + M_{p,6} T f(x) + T_* f(x)). \end{aligned} \quad (43)$$

In addition, for all doubling balls  $Q \subset R$  with  $x \in Q$  such that  $K_{Q,R} \leq P_0$  where  $P_0$  is a constant in Lemma 7.5, by (40) we have

$$|h_Q - h_R| \leq C \|b\|_{\text{RBMO}(\mu)} (M_{p,5} f(x) + T_* f(x)) P_0^2.$$

Due to Lemma 7.5 we get

$$|h_Q - h_R| \leq C \|b\|_{\text{RBMO}(\mu)} (M_{p,5} f(x) + T_* f(x)) K_{Q,R},$$

for all doubling balls  $Q \subset R$  with  $x \in Q$ . At this stage, applying (41), we obtain

$$\begin{aligned} &m_Q([b, T]f) - m_R([b, T]f) \\ &\leq C \|b\|_{\text{RBMO}(\mu)} (M_{p,5} f(x) + M_{p,6} T f(x) + T_* f(x)) K_{Q,R}. \end{aligned}$$

This completes our proof.

**Remark 7.7** *As mentioned earlier in this paper, the results of this article still hold when  $X$  is a quasi-metric space. Indeed, one can see that the main problem in quasi-metric space setting is that the covering lemma, Lemma 2.1, may not be true. However, instead of using this covering property, we can adapt the covering lemma in [FGL, Lemma 3.1] to our situation. This problem is not difficult and we leave it to the interested reader.*

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